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Double Inertial Relaxed Projection Algorithm for Solving Quasimonotone Uniformly Continuous Variational Inequality Problems

Xinyi Wei, Hongwei Liu and Meiying Wang*

(*School of Mathematics and Statistics, Xidian University, Xi'an 710126, China*)

Abstract: In this study, an original projection-based algorithm is proposed to treat a family of variational inequality models defined on a Hilbert space. The method extends Tseng's extragradient framework by integrating a double inertial mechanism and a relaxation technique. Provided that quasimonotonicity and uniform continuity hold, the sequence yielded by the algorithm is shown to converge weakly. In contrast to previous studies on quasimonotone variational inequality problems, this study requires neither the assumption of Lipschitz continuity for the mapping nor the assumption that the mapping A satisfies $Ax \neq 0$. Thus, this study addresses a more general category of quasimonotone variational inequality problems. In the final stage, numerical experiments are conducted to showcase the effectiveness and comparative advantages of the proposed method.

Keywords: projected gradient algorithm; quasimonotone mapping; uniformly continuous; variational inequality problem

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0 Introduction

This work primarily focuses on a class of iterative methods rooted in projection techniques, which are developed to resolve variational inequalities in the context of real Hilbert spaces. Let H be a Hilbert space over the real field. Within H, we consider a set C that satisfies three conditions: it is nonempty, closed and convex. Furthermore, we define an operator A that maps every element of C to an element in H. In this context, the symbols $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are used to represent the inner product operation and the induced norm within H, in that order.

The central task of the Variational Inequality Problem (VIP) involves identifying an element, denoted here as $t \in C$ such that

$$\langle At, h - t \rangle \geq 0, \forall h \in C$$

The collection of all such valid elements t that solve the VIP is represented by the symbol S, i.e.,

$$S = \{t \in C: \langle At, h - t \rangle \geq 0, \forall h \in C\}$$

Correspondingly, the Dual Variational Inequality Problem (DVIP) is defined as finding a point $t \in C$ such that

$$\langle Ah, h - t \rangle \geq 0, \forall h \in C$$

This set of solutions for the DVIP is denoted as S_D, i.e., $S_D = \{t \in C: \langle Ah, h - t \rangle \geq 0, \forall h \in C\}$. Under the conditions of a continuous mapping A and a convex set C, Theorem 2.1 in Ref. [1] established the relation $S_D \subseteq S$. When the mapping A is pseudomonotonic and continuous, the conclusion $S_D = S$ holds. However, for quasimonotonic mappings A, $S \subseteq S_D$ does not always apply.

VIP provides a unified mathematical framework[2-3] for studying a broad range of equilibrium problems arising from optimization, economics, engineering, and scientific computing[4]. Its theoretical foundations and methodological tools have thereby become indispensable for analyzing numerous complex systems. Due to the widespread application of VIP, numerous researchers[5-10] have developed methods to address VIP and enhance the convergence rate of these methods, enabling faster attainment of the VIP solution. The projected gradient method represents the traditional iterative technique and can be described as follows:

$$t_{k+1} = P_C(t_k - \lambda A t_k)$$

where $\lambda \in (0, L^{-1})$ and the symbol L is used to

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* Corresponding author. Hongwei Liu, Ph.D, Professor. Email: hwliuxidian@163.com.

characterize the Lipschitz continuity of the operator A. When the operator A enjoys strong monotonicity and Lipschitz continuity, it was demonstrated in Ref.[11] that the iterates $\{t_k\}$ produced by the algorithm converge to the solution of VIP. However, when A is merely monotone, the sequence $\{t_k\}$ produced by this algorithm fails to converge to the solution. To address this issue, Korpelevich[12] introduced the extragradient algorithm for solving VIP in 1977. The iterative scheme is as follows:

$$\begin{cases} h_k = P_C(t_k - \lambda A t_k) \\ t_{k+1} = P_C(t_k - \lambda A h_k) \end{cases}$$

Under the condition that the operator A exhibits both monotonicity and Lipschitz continuity, with a parameter $\lambda \in (0, L^{-1})$. This algorithm also has certain limitations, as it requires performing two projection operations during each iteration. For problems with a structurally complex feasible set C, projecting onto C often entails high computational cost, thereby limiting practical applicability. To address this issue, Censor[13] proposed a subgradient extragradient method, with the iterative process as follows:

$$\begin{cases} h_k = P_C(t_k - \lambda A t_k) \\ T_k = \{t \in H: \langle t_k - \lambda A t_k - h_k, t - h_k \rangle \leqslant 0\} \\ t_{k+1} = P_{T_k}(t_k - \lambda A h_k) \end{cases}$$

Subsequently, Tseng[14] introduced a modified extragradient algorithm to enhance the computational efficiency for solving VIP. The iterative scheme is presented as follows:

$$\begin{cases} h_k = P_C(t_k - \lambda A t_k) \\ t_{k+1} = h_k + \lambda (A t_k - A h_k) \end{cases}$$

All the algorithms mentioned above employ fixed step sizes. Although they can obtain VIP solutions, their computational efficiency remains limited. Consequently, building upon these foundational algorithms, researchers have explored the adoption of adaptive step size strategies. For instance, Cai et al.[15] proposed the following algorithm in 2021:

$$\begin{cases} h_k = P_C(t_k - \lambda_k A t_k) \\ z_k = P_{T_k}(t_k - \lambda_k A h_k) \\ t_{k+1} = \beta_k f(t_k) + (1 - \beta_k) z_k \end{cases}$$

The algorithm step size λ_k is the maximum value of $\lambda \in \{\gamma, \gamma l, \gamma l^2, \cdots\}$ satisfying the condition as follows:

$$\lambda \langle A h_k - A t_k, h_k - z_k \rangle \leqslant \frac{\mu}{2} [\, \| t_k - h_k \|^2 + \| h_k - z_k \|^2]$$

and

$$T_k := \{t \in H \mid \langle t_k - \lambda_k A t_k - h_k, t - h_k \rangle \leqslant 0\}$$

For an operator A that is both pseudomonotone and fails to be Lipschitz continuous, the iteration sequence $\{t_k\}$ produced by the method can be demonstrated to possess the property of strong convergence. Meanwhile, researchers have considered incorporating an inertial term into the algorithm to improve its solving efficiency[16-18]. For example, in 2022, Yao et al.[19] presented an extragradient-type subgradient method with two inertial terms for solving VIP. The algorithm format is as follows:

$$\begin{cases} z_k = t_k + \delta (t_k - t_{k-1}) \\ w_k = t_k + \theta_k (t_k - t_{k-1}) \\ h_k = P_C(w_k - \lambda_k A w_k) \\ t_{k+1} = (1 - \alpha_k) z_k + \alpha_k P_{T_k}(w_k - \lambda_k A h_k) \end{cases}$$

Within the framework of this algorithmic approach, strong convergence has been established for the iterative sequence $\{t_k\}$. This result holds under the key assumption that the operator A exhibits the property of strong pseudomonotonicity. Additionally, under appropriate parameter assumptions, the algorithm is demonstrated to achieve a linear convergence rate.

This work introduces an enhanced version of Tseng's extragradient method. A variable step-size mechanism determined by an Armijo-type line search constitutes the key modification, replacing the original fixed step-size scheme. Inspired by Ref. [19], a double inertial component is employed within the algorithm to enhance the rate at which the iterative sequence converges. Additionally, we consider a quasimonotone VIP in a Hilbert space where the mapping A is non-Lipschitz continuous. Unlike the conventional assumption for quasimonotone variational inequalities, which requires the mapping A to satisfy condition $At \neq 0$, $\forall t \in H$, the quasimonotonic variational inequality problem discussed in this paper encompasses this case[20]. Under appropriate theoretical conditions, this research establishes that the iterative points $\{\boldsymbol{v}_k\}$, which are produced by the proposed procedure, exhibit weak convergence.

1 Preliminaries

Key definitions and lemmas relevant to the proofs in the following sections are discussed in this section. The symbols $\rightharpoonup$ and $\rightarrow$ are adopted in this work to signify the weak and strong convergence, respectively, for the sequence $\{v_\tau\}$, i.e., $v_\tau \rightharpoonup v$ and $v_\tau \to v$. For any $\omega, \iota \in H$ and $\alpha, \beta \in \mathbb{R}$, then we have

$$\|\alpha\omega + \beta\iota\|^2 = \alpha(\alpha+\beta)\|\omega\|^2 + \beta(\alpha+\beta)\|\iota\|^2 - \alpha\beta\|\omega - \iota\|^2 \quad (1)$$

Corresponding to every element $t \in H$, one can find a single closest element $P_C(t) \in C$ which meets

$$P_C(t) := \arg\min_{h \in C}\{\|t - h\|\}$$

where $P_C(t)$ denotes the projection of point t onto the set C, and this projection mapping is nonexpansive.

Definition 1 Given a mapping $A: H \to H$ be a mapping, we describe A as having the following properties:

1) The operator A satisfies the L-Lipschitz condition on H provided that the inequality below holds for some constant $L > 0$

$$\|A\omega - A\iota\| \leqslant L\|\omega - \iota\|, \forall \omega, \iota \in H$$

2) It is η-strongly monotone on H whenever the inequality below holds for some constant $\eta > 0$,

$$\langle A\omega - A\iota, \omega - \iota\rangle \geqslant \eta\|\omega - \iota\|^2, \forall \omega, \iota \in H$$

3) Monotone on H, if

$$\langle A\omega - A\iota, \omega - \iota\rangle \geqslant 0, \forall \omega, \iota \in H$$

4) μ-strongly pseudomonotone on H, if there exists a constant $\mu > 0$ such that

$$\langle A\omega, \iota - \omega\rangle \geqslant 0 \Rightarrow \langle A\iota, \omega - \iota\rangle \geqslant \mu\|\omega - \iota\|^2, \forall \omega, \iota \in H$$

5) Pseudomonotone on H, if

$$\langle A\omega, \iota - \omega\rangle \geqslant 0 \Rightarrow \langle A\iota, \iota - \omega\rangle \geqslant 0, \forall \omega, \iota \in H$$

6) Quasimonotone on H, if

$$\langle A\omega, \iota - \omega\rangle > 0 \Rightarrow \langle A\iota, \iota - \omega\rangle \geqslant 0, \forall \omega, \iota \in H$$

Based on the above definitions, we have the conclusion that: 2) ⇒ 3) ⇒ 5) ⇒ 6) and 4) ⇒ 5) ⇒ 6), nevertheless, the conclusion cannot be deduced in the opposite direction.

Lemma 1[21] Let the sequences $\{p_\tau\}$, $\{q_\tau\}$ and $\{\chi_\tau\}$ be defined over an interval $[0, +\infty)$, satisfying

$$p_{\tau+1} \leqslant p_\tau + \chi_\tau(p_\tau - p_{\tau-1}) + q_\tau, \forall \tau \geqslant 1$$

where the sequence $\{q_\tau\}$ satisfies $\sum_{\tau=1}^{+\infty} q_\tau < +\infty$ and there is a constant $\chi \in (0,1)$ for which $0 \leqslant \chi_\tau \leqslant \chi < 1$. If it holds true for any arbitrary $\tau \in \mathbb{N}$, then $\lim_{\tau\to\infty} p_\tau = p^*$ and $p^* \in [0, +\infty)$.

Lemma 2[22] Consider a sequence $\{u_\tau\}$ taking values in the underlying space H, and assume that u_n weakly converges to u. Then we have:

$$\liminf_{\tau\to\infty}\|u_\tau - u\| < \liminf_{\tau\to\infty}\|u_\tau - v\|, \forall v \neq u$$

Lemma 3[23] Consider a nonempty closed convex subset $C \subseteq H$. For any $u \in H$, we have

$$\langle P_C(u) - u, v - P_C(u)\rangle \geqslant 0, \forall v \in C$$

Lemma 4[24] Let H_1 and H_2 be two real Hilbert spaces. Suppose an operator $A: H_1 \to H_2$ exhibits uniform continuity when restricted to any bounded domain within H_1. If D represents a bounded set in H_1, it follows that its image $A(D)$ under the mapping is likewise bounded.

Lemma 5[25] For any $\varpi \in H$ and $\alpha \geqslant \beta > 0$, the following inequality holds:

$$\frac{\|\varpi - P_C(\varpi - \alpha A\varpi)\|}{\alpha} \leqslant \frac{\|\varpi - P_C(\varpi - \beta A\varpi)\|}{\beta}$$

$$\|\varpi - P_C(\varpi - \beta A\varpi)\| \leqslant \|\varpi - P_C(\varpi - \alpha A\varpi)\|$$

2 Algorithm and Convergence Analysis

We start by presenting the basic assumptions needed for the analysis, then verify the weak convergence of the generated sequence.

Assumptions:

(A) $S_D \neq \varnothing$.

(B) A is a quasimonotone operator, uniformly continuous on a real Hilbert space H and sequentially weakly continuous on the set C. (If a sequence $\{v_\tau\} \subseteq H$ and $v_\tau \rightharpoonup v$, then $Av_\tau \rightharpoonup Av$).

(C) If $v_\tau \rightharpoonup v$, then $\liminf_{\tau\to\infty}\|Av_\tau\| \geqslant \|Av\|$.

(D) $T = \{u \in C: Au = 0\} \backslash S_D$ is a finite set.

(E) The following are the parameter settings required for the inertial term and relaxation term in the algorithm:

i. $0 \leqslant \sigma < \min\{\frac{\beta - \sqrt{2\beta}}{\beta}, \theta_1\}, \beta \in (2, \infty)$;

ii. $0 \leqslant \theta_\tau \leqslant \theta_{\tau+1} \leqslant 1$;

iii. $0 < \xi \leqslant \xi_\tau \leqslant \xi_{\tau+1} < \frac{1}{1+\beta}, \beta \in (2, \infty)$.

Algorithm 1: Double inertial relaxed Tseng-type extragradient algorithm

Initialization: Given parameters $\gamma > 0, l \in (0,1)$ and $\mu \in (0,1)$ Let $\lambda_0 > 0$ and u_0, $u_1 \in H$ be chosen arbitrarily.

Step 1: Given $v_{\tau-1}$ and $v_\tau (\tau \geqslant 1)$, compute

$$\begin{cases} p_\tau = v_\tau + \sigma(v_\tau - v_{\tau-1}) \\ \hbar_\tau = v_\tau + \theta_\tau(v_\tau - v_{\tau-1}) \end{cases}$$

Step 2: Compute

$$s_\tau = P_C(\hbar_\tau - \lambda_\tau A\hbar_\tau)$$

$$\omega_\tau = s_\tau + \lambda_\tau(A\hbar_\tau - As_\tau)$$

Step 3: Calculate

$$v_{\tau+1} = (1-\xi_\tau)p_\tau + \xi_\tau\omega_\tau$$

and update the step size, setting $\lambda_\tau = \gamma l^{m_\tau}$ and the value of m_τ is determined as the least positive integer m ensuring that the inequality presented below holds true:

$$\gamma l^m \|A\hbar_\tau - As_\tau\| \leqslant \mu\|\hbar_\tau - s_\tau\| \quad (2)$$

Set $\tau := \tau + 1$, and proceed to Step 1

Lemma 6[26] Under the assumption that conditions (A)−(E) hold, the adaptive line search step satisfying inequality (2) is rigorously defined.

The proof follows a similar argument to that of Lemma 8 in Ref.[26], see therein for details.

Lemma 7 Assuming conditions (A)−(E) hold, the sequence $\{v_\tau\}$ produced by Algorithm 1 remains bounded and satisfies $\lim_{\tau\to\infty}\|v_{\tau+1} - v_\tau\| = 0$. Moreover, for any $\varpi \in S_D$, $\lim_{\tau\to\infty}\|v_\tau - \varpi\|$ exists.

Proof Arbitrarily select a point $\varpi \in S_D$, we have

$$\begin{aligned}\|\omega_\tau - \varpi\|^2 &= \|s_\tau + \lambda_\tau(A\hbar_\tau - As_\tau) - \varpi\|^2 = \\ &\|s_\tau - \varpi\|^2 + \lambda_\tau^2\|A\hbar_\tau - As_\tau\|^2 + 2\lambda_\tau\langle s_\tau - \varpi, A\hbar_\tau - As_\tau\rangle = \|s_\tau + \hbar_\tau - \hbar_\tau - \varpi\|^2 + \\ &\lambda_\tau^2\|A\hbar_\tau - As_\tau\|^2 + 2\lambda_\tau\langle s_\tau - \varpi, A\hbar_\tau - As_\tau\rangle = \\ &\|\hbar_\tau - \varpi\|^2 + \|s_\tau - \hbar_\tau\|^2 + 2\langle s_\tau - \hbar_\tau, \hbar_\tau - \varpi\rangle + \lambda_\tau^2\|A\hbar_\tau - As_\tau\|^2 + 2\lambda_\tau\langle s_\tau - \varpi, A\hbar_\tau - As_\tau\rangle\end{aligned} \quad (3)$$

Given the definition $s_\tau = P_C(\hbar_\tau - \lambda_\tau A\hbar_\tau)$, the ensuing derivation leverages fundamental properties of the projection operator.

$$\langle \hbar_\tau - \lambda_\tau A\hbar_\tau - s_\tau, \varpi - s_\tau\rangle \leqslant 0$$

Consequently, we get

$$\langle s_\tau - \hbar_\tau, s_\tau - \varpi\rangle \leqslant \lambda_\tau\langle A\hbar_\tau, \varpi - s_\tau\rangle \quad (4)$$

From inequality (3) and Eq.(4), we get

$$\begin{aligned}\|\omega_\tau - \varpi\|^2 &= \|\hbar_\tau - \varpi\|^2 + \|s_\tau - \hbar_\tau\|^2 + 2\langle s_\tau - \hbar_\tau, s_\tau - \varpi\rangle + 2\langle s_\tau - \hbar_\tau, \hbar_\tau - s_\tau\rangle + \lambda_\tau^2\|A\hbar_\tau - As_\tau\|^2 + 2\lambda_\tau\langle s_\tau - \varpi, A\hbar_\tau - As_\tau\rangle \leqslant \|\hbar_\tau - \\ &\varpi\|^2 - \|s_\tau - \hbar_\tau\|^2 + \lambda_\tau^2\|A\hbar_\tau - As_\tau\|^2 + 2\lambda_\tau\langle A\hbar_\tau, \varpi - s_\tau\rangle + 2\lambda_\tau\langle s_\tau - \varpi, A\hbar_\tau - As_\tau\rangle = \\ &\|\hbar_\tau - \varpi\|^2 - \|s_\tau - \hbar_\tau\|^2 + \lambda_\tau^2\|A\hbar_\tau - As_\tau\|^2 + 2\lambda_\tau\langle As_\tau, \varpi - s_\tau\rangle\end{aligned}$$

Since $\varpi \in S_D$, we have

$$\langle As_\tau, \varpi - s_\tau\rangle \leqslant 0 \quad (5)$$

Therefore, by the definition of step size λ_τ, inequality (5) and $\mu \in (0,1)$, we derive the following inequality chain:

$$\begin{aligned}\|\omega_\tau - \varpi\|^2 &\leqslant \|\hbar_\tau - \varpi\|^2 - \|s_\tau - \hbar_\tau\|^2 + \lambda_\tau^2\|A\hbar_\tau - As_\tau\|^2 \leqslant \|\hbar_\tau - \varpi\|^2 - \|s_\tau - \\ &\hbar_\tau\|^2 + \mu^2\|\hbar_\tau - s_\tau\|^2 = \|\hbar_\tau - \varpi\|^2 - (1-\mu^2)\|\hbar_\tau - s_\tau\|^2 \leqslant \|\hbar_\tau - \varpi\|^2\end{aligned} \quad (6)$$

From the definition $v_{\tau+1}$ in Algorithm 1 and inequality (6), we can obtain:

$$\begin{aligned}\|v_{\tau+1} - \varpi\|^2 &= \|(1-\xi_\tau)p_\tau + \xi_\tau\omega_\tau - \varpi\|^2 = \\ &(1-\xi_\tau)\|p_\tau - \varpi\|^2 + \xi_\tau\|\omega_\tau - \varpi\|^2 - \xi_\tau(1-\xi_\tau)\|p_\tau - \omega_\tau\|^2 \leqslant (1-\xi_\tau)\|p_\tau - \varpi\|^2 + \\ &\xi_\tau\|\hbar_\tau - \varpi\|^2 - \xi_\tau(1-\xi_\tau)\|p_\tau - \omega_\tau\|^2\end{aligned} \quad (7)$$

According to $v_{\tau+1} = (1-\xi_\tau)p_\tau + \xi_\tau\omega_\tau$, there is

$$\|p_\tau - \omega_\tau\| = \frac{\|v_{\tau+1} - p_\tau\|}{\xi_\tau}$$

Substituting the above expression into inequality (7), we obtain

$$\|v_{\tau+1} - \varpi\|^2 \leqslant (1-\xi_\tau)\|p_\tau - \varpi\|^2 + \xi_\tau\|\hbar_\tau - \varpi\|^2 - \frac{(1-\xi_\tau)}{\xi_\tau}\|v_{\tau+1} - p_\tau\|^2 \quad (8)$$

By the definitions of p_τ and $\hbar_\tau$, and Eq.(1), we have

$$\begin{aligned}\|p_\tau - \varpi\|^2 &= \|v_\tau + \sigma(v_\tau - v_{\tau-1}) - \varpi\|^2 = \\ &\|(1+\sigma)(v_\tau - \varpi) - \sigma(v_{\tau-1} - \varpi)\|^2 = \\ &(1+\sigma)\|v_\tau - \varpi\|^2 - \sigma\|v_{\tau-1} - \varpi\|^2 + \\ &\sigma(1+\sigma)\|v_\tau - v_{\tau-1}\|^2\end{aligned} \quad (9)$$

and

$$\begin{aligned}\|\hbar_\tau - \varpi\|^2 &= \|v_\tau + \theta_\tau(v_\tau - v_{\tau-1}) - \varpi\|^2 = \\ &\|(1+\theta_\tau)(v_\tau - \varpi) - \theta_\tau(v_{\tau-1} - \varpi)\|^2 = \\ &(1+\theta_\tau)\|v_\tau - \varpi\|^2 - \theta_\tau\|v_{\tau-1} - \varpi\|^2 + \\ &\theta_\tau(1+\theta_\tau)\|v_\tau - v_{\tau-1}\|^2\end{aligned} \quad (10)$$

and

$$\begin{aligned}\|v_{\tau+1} - p_\tau\|^2 &= \|v_{\tau+1} - v_\tau - \sigma(v_\tau - v_{\tau-1})\|^2 = \\ &\|v_{\tau+1} - v_\tau\|^2 + \sigma^2\|v_\tau - v_{\tau-1}\|^2 - 2\sigma\langle v_{\tau+1} - v_\tau, v_\tau - v_{\tau-1}\rangle \geqslant \|v_{\tau+1} - v_\tau\|^2 + \\ &\sigma^2\|v_\tau - v_{\tau-1}\|^2 - 2\sigma\|v_{\tau+1} - v_\tau\|\|v_\tau - v_{\tau-1}\| \geqslant (1-\sigma)\|v_{\tau+1} - v_\tau\|^2 + (\sigma^2 - \\ &\sigma)\|v_\tau - v_{\tau-1}\|^2\end{aligned} \quad (11)$$

Substituting Eqs. (9), (10) and (11) into inequality (8), we obtain

$$\begin{aligned}\|v_{\tau+1} - \varpi\|^2 &\leqslant (1-\xi_\tau)[(1+\sigma)\|v_\tau - \varpi\|^2 - \sigma\|v_{\tau-1} - \varpi\|^2 + \sigma(1+\sigma)\|v_\tau - v_{\tau-1}\|^2] + \\ &\xi_\tau[(1+\theta_\tau)\|v_\tau - \varpi\|^2 - \theta_\tau\|v_{\tau-1} - \varpi\|^2 + \theta_\tau(1+\theta_\tau)\|v_\tau - v_{\tau-1}\|^2] - \frac{1-\xi_\tau}{\xi_\tau}[(1- \\ &\sigma)\|v_{\tau+1} - v_\tau\|^2 + (\sigma^2 - \sigma)\|v_\tau - v_{\tau-1}\|^2] = \\ &[(1-\xi_\tau)(1+\sigma) + \xi_\tau(1+\theta_\tau)]\|v_\tau - \varpi\|^2 -\end{aligned}$$

$[(1-\xi_\tau)\sigma+\theta_\tau\xi_\tau]\|v_{\tau-1}-\varpi\|^2+[(1-\xi_\tau)\sigma(1+\sigma)+\xi_\tau\theta_\tau(1+\theta_\tau)-\frac{1-\xi_\tau}{\xi_\tau}(\sigma^2-\sigma)]\|v_\tau-v_{\tau-1}\|^2-\frac{1-\xi_\tau}{\xi_\tau}(1-\sigma)\|v_{\tau+1}-v_\tau\|^2$

where we define α_τ and β_τ as

$$\alpha_\tau=\frac{1-\xi_\tau}{\xi_\tau}(1-\sigma)$$

$$\beta_\tau=(1-\xi_\tau)\sigma(1+\sigma)+\xi_\tau\theta_\tau(1+\theta_\tau)-\frac{1-\xi_\tau}{\xi_\tau}(\sigma^2-\sigma)$$

Therefore, we can obtain:

$$\|v_{\tau+1}-\varpi\|^2\leqslant(1+\xi_\tau\theta_\tau+\sigma(1-\xi_\tau))\|v_\tau-\varpi\|^2-(\xi_\tau\theta_\tau+\sigma(1-\xi_\tau))\|v_{\tau-1}-\varpi\|^2-\alpha_\tau\|v_{\tau+1}-v_\tau\|^2+\beta_\tau\|v_\tau-v_{\tau-1}\|^2 \quad (12)$$

Next, we define

$$\Gamma_\tau=\|v_\tau-\varpi\|^2-(\xi_\tau\theta_\tau+\sigma(1-\xi_\tau))\|v_{\tau-1}-\varpi\|^2+\beta_\tau\|v_\tau-v_{\tau-1}\|^2$$

From inequlity (12), we have

$$\Gamma_{\tau+1}-\Gamma_\tau=\|v_{\tau+1}-\varpi\|^2-(\xi_{\tau+1}\theta_{\tau+1}+\sigma(1-\xi_{\tau+1}))\|v_\tau-\varpi\|^2+\beta_{\tau+1}\|v_{\tau+1}-v_\tau\|^2-\|v_\tau-\varpi\|^2+(\xi_\tau\theta_\tau+\sigma(1-\xi_\tau))\|v_{\tau-1}-\varpi\|^2-\beta_\tau\|v_\tau-v_{\tau-1}\|^2\leqslant(\xi_\tau\theta_\tau+\sigma(1-\xi_\tau))-\xi_{\tau+1}\theta_{\tau+1}-\sigma(1-\xi_{\tau+1}))\|v_\tau-\varpi\|^2-\alpha_\tau\|v_{\tau+1}-v_\tau\|^2+\beta_{\tau+1}\|v_{\tau+1}-v_\tau\|^2=((\theta_\tau-\sigma)\xi_\tau-(\theta_{\tau+1}-\sigma)\xi_{\tau+1})\|v_\tau-\varpi\|^2-\alpha_\tau\|v_{\tau+1}-v_\tau\|^2+\beta_{\tau+1}\|v_{\tau+1}-v_\tau\|^2 \quad (13)$$

According to $\theta_\tau\leqslant\theta_{\tau+1}$, $\xi_\tau\leqslant\xi_{\tau+1}$ and $0\leqslant\sigma\leqslant\theta_1\leqslant\theta_\tau$, it holds for all $\tau\geqslant 0$. $\theta_\tau-\sigma\geqslant 0$ and $\theta_{\tau+1}-\sigma\geqslant 0$ can be obtained. Subsequently, we can obtain $\theta_\tau-\sigma\leqslant\theta_{\tau+1}-\sigma$ and $(\theta_\tau-\sigma)\xi_\tau\leqslant(\theta_{\tau+1}-\sigma)\xi_{\tau+1}$. Therefore, inequality (13) can be expressed as follows:

$$\Gamma_{\tau+1}-\Gamma_\tau\leqslant-\alpha_\tau\|v_{\tau+1}-v_\tau\|^2+\beta_{\tau+1}\|v_{\tau+1}-v_\tau\|^2\leqslant-(\alpha_\tau-\beta_{\tau+1})\|v_{\tau+1}-v_\tau\|^2 \quad (14)$$

Due to $\xi_\tau<\frac{1}{1+\beta}$ and $\theta_\tau,\sigma\leqslant 1$, we have

$$\alpha_\tau-\beta_{\tau+1}=\frac{1-\xi_\tau}{\xi_\tau}(1-\sigma)-[(1-\xi_{\tau+1})\sigma(1+\sigma)+\xi_{\tau+1}\theta_{\tau+1}(1+\theta_{\tau+1})-\frac{1-\xi_{\tau+1}}{\xi_{\tau+1}}(\sigma^2-\sigma)]\geqslant\beta(1-\sigma)+\beta(\sigma^2-\sigma)-(1-\xi_{\tau+1})\sigma(1+\sigma)-2\xi_{\tau+1}>\beta(1-\sigma)+\beta(\sigma^2-\sigma)-2(1-\xi_{\tau+1})-2\xi_{\tau+1}>\beta(1-\sigma)+\beta(\sigma^2-\sigma)-2=\beta\sigma^2-2\beta\sigma+\beta-2 \quad (15)$$

From Assumption (E), we have $\sigma<\frac{\beta-\sqrt{2\beta}}{\beta}$ and furthermore we can obtain $\zeta:=\beta\sigma^2-2\beta\sigma+\beta-2>0$. Therefore, from inequalities (14) and (15), we obtain

$$\Gamma_{\tau+1}-\Gamma_\tau\leqslant-\zeta\|v_{\tau+1}-v_\tau\|^2\leqslant 0 \quad (16)$$

Therefore, it follows that the sequence $\{\Gamma_\tau\}$ does not increase. Based on the definition of Γ_τ, we consequently derive

$$\Gamma_{\tau+1}\geqslant-(\xi_{\tau+1}\theta_{\tau+1}+\sigma(1-\xi_{\tau+1}))\|v_\tau-\varpi\|^2 \quad (17)$$

and

$$\Gamma_\tau\geqslant\|v_\tau-\varpi\|^2-(\xi_\tau\theta_\tau+\sigma(1-\xi_\tau))\|v_{\tau-1}-\varpi\|^2 \quad (18)$$

From $0<\xi\leqslant\xi_\tau<\frac{1}{1+\beta}$, we can obtain that,

$$\begin{aligned}\|v_\tau-\varpi\|^2&\leqslant(\xi_\tau\theta_\tau+\sigma(1-\xi_\tau))\|v_{\tau-1}-\varpi\|^2+\Gamma_\tau\leqslant(\frac{1}{1+\beta}+\sigma(1-\xi))\|v_{\tau-1}-\varpi\|^2+\Gamma_1=\varphi\|v_{\tau-1}-\varpi\|^2+\Gamma_1\\&\vdots\\&\leqslant\varphi^\tau\|v_0-\varpi\|^2+(1+\varphi+\cdots+\varphi^{\tau-1})\Gamma_1\\&\leqslant\varphi^\tau\|v_0-\varpi\|^2+\frac{\Gamma_1}{1-\varphi}\leqslant\|v_0-\varpi\|^2+\frac{\Gamma_1}{1-\varphi}\end{aligned} \quad (19)$$

where $\varphi:=(1+\beta)^{-1}+\sigma(1-\xi)<1$, this is because $\sigma<\frac{\beta-\sqrt{2\beta}}{\beta}<\frac{\beta}{(1+\beta)(1-\xi)}$, $\xi\in(0,1)$. From inequality (19), we have that $\{\|v_\tau-\varpi\|\}$ is bounded and $\{v_\tau\}$ is bounded.

We now prove that $\lim\limits_{\tau\to\infty}\|v_{\tau+1}-v_\tau\|=0$. From inequalities (17) and (19), we have

$$-\Gamma_{\tau+1}\leqslant(\xi_{\tau+1}\theta_{\tau+1}+\sigma(1-\xi_{\tau+1}))\|v_\tau-\varpi\|^2\leqslant(\frac{1}{1+\beta}+\sigma(1-\xi))\|v_\tau-\varpi\|^2=\varphi\|v_\tau-\varpi\|^2\leqslant\varphi^{\tau+1}\|v_0-\varpi\|^2+\frac{\varphi\Gamma_1}{1-\varphi} \quad (20)$$

The following result can be derived from inequalities (16) and (20):

$$\zeta\sum_{r=1}^{\tau}\|v_{r+1}-v_r\|^2\leqslant\sum_{r=1}^{\tau}(\Gamma_r-\Gamma_{r+1})=\Gamma_1-\Gamma_{\tau+1}\leqslant\frac{\Gamma_1}{1-\varphi}+\varphi^{\tau+1}\|v_0-\varpi\|^2$$

This leads to the establishment of $\sum\limits_{\tau=1}^{\infty}\|v_{\tau+1}-$

$\upsilon_\tau \|^2 \leqslant \dfrac{\Gamma_1}{\zeta(1-\varphi)} < +\infty$, confirming the convergence of the series $\sum_{\tau=1}^{\infty} \| \upsilon_{\tau+1} - \upsilon_\tau \|^2$. Consequently, it can be concluded that $\lim_{\tau\to\infty} \| \upsilon_{\tau+1} - \upsilon_\tau \| = 0$.

We proceed to show that $\lim_{\tau\to\infty} \| \upsilon_\tau - \varpi \|$ exists for any ϖ in S_D. From inequalitiy (12), we have

$$\| \upsilon_{\tau+1} - \varpi \|^2 \leqslant \| \upsilon_\tau - \varpi \|^2 + (\xi_\tau + \sigma(1 - \xi_\tau))(\| \upsilon_\tau - \varpi \|^2 - \| \upsilon_{\tau-1} - \varpi \|^2) + \beta_\tau \| \upsilon_\tau - \upsilon_{\tau-1} \|^2 \quad (21)$$

where $\beta_\tau \leqslant (1-\xi)\sigma(1+\sigma) + \dfrac{2}{1+\beta} + \dfrac{1-\xi}{\xi}(\sigma - \sigma^2)$ and $\xi_\tau + \sigma(1-\xi_\tau) < \dfrac{1}{1+\beta} + \sigma(1-\xi) < 1$.

Therefore, based on $\sum_{\tau=1}^{\infty} \| \upsilon_{\tau+1} - \upsilon_\tau \|^2 \leqslant \dfrac{\Gamma_1}{\zeta(1-\varphi)} < +\infty$, inequality (21) and Lemma 1, we conclude that $\lim_{\tau\to\infty} \| \upsilon_\tau - \varpi \|$ exists for any ϖ in S_D.

Lemma 8 Under Assumptions (A) - (E), consider the sequence $\{\upsilon_\tau\}$ constructed through Algorithm 1. The ensuing limits are established:

$$\lim_{\tau\to\infty} \| s_\tau - \hbar_\tau \| = 0$$
$$\lim_{\tau\to\infty} \| \upsilon_\tau - \hbar_\tau \| = 0$$
$$\lim_{\tau\to\infty} \| \upsilon_\tau - s_\tau \| = 0$$

Proof Based on Algorithm 1 and $\lim_{\tau\to\infty} \| \upsilon_{\tau+1} - \upsilon_\tau \| = 0$ in Lemma 7, we have

$$\lim_{\tau\to\infty} \| \upsilon_\tau - \hbar_\tau \| = \lim_{\tau\to\infty} \| \upsilon_\tau - [\upsilon_\tau + \theta_\tau(\upsilon_\tau - \upsilon_{\tau-1})] \| = \lim_{\tau\to\infty} \theta_\tau \| \upsilon_\tau - \upsilon_{\tau-1} \| = 0$$

Below, we demonstrate that sequences $\{\hbar_\tau\}$ and $\{\omega_\tau\}$ are bounded. Since $\lim_{\tau\to\infty} \| \upsilon_\tau - \hbar_\tau \| = 0$ and $\{\upsilon_\tau\}$ is bounded, $\{\hbar_\tau\}$ can be seen to be bounded as well. Meanwhile, applying the triangle inequality, we get the inequality as follows:

$$\| \hbar_\tau - \omega_\tau \| \leqslant \| \hbar_\tau - \upsilon_\tau \| + \| \upsilon_\tau - p_\tau \| + \| p_\tau - \omega_\tau \| \quad (22)$$

where $\lim_{\tau\to\infty} \| \upsilon_\tau - p_\tau \| = \lim_{\tau\to\infty} \sigma \| \upsilon_\tau - \upsilon_{\tau-1} \| = 0$ and

$$\lim_{\tau\to\infty} \| p_\tau - \omega_\tau \| = \lim_{\tau\to\infty} \frac{\| \upsilon_{\tau+1} - p_\tau \|}{\xi_\tau} \leqslant \lim_{\tau\to\infty} \frac{\| \upsilon_{\tau+1} - \upsilon_\tau \| + \| \upsilon_\tau - p_\tau \|}{\xi_\tau} = 0$$

Taking the limit on both sides of inequality (22), we obtain $\lim_{\tau\to\infty} \| \hbar_\tau - \omega_\tau \| = 0$. As a result, the sequence $\{\omega_\tau\}$ is bounded. We now prove that $\lim_{\tau\to\infty} \| s_\tau - \hbar_\tau \| = 0$. From inequality (6), we have the inequality as follows:

$$(1-\mu^2) \| s_\tau - \hbar_\tau \|^2 \leqslant \| \hbar_\tau - \varpi \|^2 - \| \omega_\tau - \varpi \|^2$$

where $1 - \mu^2 > 0$. With the sequences $\{\hbar_\tau\}$ and $\{\omega_\tau\}$ being bounded, one can find a positive constant $K > 0$ for which

$$(1-\mu^2) \| s_\tau - \hbar_\tau \|^2 \leqslant \| \hbar_\tau - \varpi \|^2 - \| \omega_\tau - \varpi \|^2 \leqslant (\| \hbar_\tau - \varpi \| + \| \omega_\tau - \varpi \|)(\| \hbar_\tau - \varpi \| - \| \omega_\tau - \varpi \|) \leqslant K \| \hbar_\tau - \omega_\tau \|$$

Taking the limit as $\tau \to \infty$, we obtain $\lim_{\tau\to\infty} \| s_\tau - \hbar_\tau \| = 0$. Finally, we prove that $\lim_{\tau\to\infty} \| \upsilon_\tau - s_\tau \| = 0$. By the triangle inequality, we have

$$\| \upsilon_\tau - s_\tau \| \leqslant \| \upsilon_\tau - \hbar_\tau \| + \| \hbar_\tau - s_\tau \|$$

Taking the limit as $\tau \to \infty$, we get $\lim_{\tau\to\infty} \| \upsilon_\tau - s_\tau \| = 0$.

Lemma 9 Under the fulfillment of conditions (A) - (E), consider the iterative sequence $\{\upsilon_\tau\}$ produced by Algorithm 1. Provided that a convergent subsequence $\{\upsilon_{\tau_k}\} \subseteq \{\upsilon_\tau\}$ exists with $\upsilon_{\tau_k} \rightharpoonup \varpi$, it follows that $\varpi \in S_D$ or $\varpi \in \{u \in C, Au = 0\}$.

Proof From Lemma 8, we have $\lim_{\tau\to\infty} \| \upsilon_\tau - s_\tau \| = 0$ and $\lim_{\tau\to\infty} \| \upsilon_\tau - \hbar_\tau \| = 0$. Since the subsequence υ_{τ_k} converges weakly to ϖ, it follows that

$$s_{\tau_k} \rightharpoonup \varpi, \ \hbar_{\tau_k} \rightharpoonup \varpi, \ \varpi \in C$$

Given that $\limsup_{k\to\infty} \| A s_{\tau_k} \| \geqslant 0$, the proof strategy involves two cases, addressing them separately:

$$\limsup_{k\to\infty} \| A s_{\tau_k} \| = 0, \quad \limsup_{k\to\infty} \| A s_{\tau_k} \| > 0$$

Case 1 Under the condition that $\limsup_{k\to\infty} \| A s_{\tau_k} \| = 0$, the following holds:

$$\lim_{k\to\infty} \| A s_{\tau_k} \| = \liminf_{k\to\infty} \| A s_{\tau_k} \| = \limsup_{k\to\infty} \| A s_{\tau_k} \| = 0$$

Given that $s_{\tau_k} \rightharpoonup \varpi$ and the mapping A satisfies Assumption (C), it follows that

$$0 \leqslant \| A \varpi \| \leqslant \liminf_{k\to\infty} \| A s_{\tau_k} \| = 0$$

which implies $A \varpi = 0$.

Case 2 The condition $\limsup_{k\to\infty} \| A s_{\tau_k} \| > 0$ guarantees the existence of a subsequence $\{A s_{\tau_{k_l}}\}$ such that $\lim_{k\to\infty} \| A s_{\tau_{k_l}} \| > 0$. For simplicity, we assume that $\lim_{k\to\infty} \| A s_{\tau_k} \| = d > 0$. Consequently, $N_1 \in \mathbb{N}$ is guaranteed to exist such that

$$\| A s_{\tau_k} \| > \frac{d}{2}, \ \forall k \geqslant N_1$$

Since $s_{\tau_k} = P_C(\hbar_{\tau_k} - \lambda_{\tau_k} A \hbar_{\tau_k})$, it follows from Lemma 3 that

$$\langle \hbar_{\tau_k} - \lambda_{\tau_k} A\hbar_{\tau_k} - s_{\tau_k}, s - s_{\tau_k} \rangle \leq 0, \forall s \in C$$

which implies that

$$\frac{1}{\lambda_{\tau_k}} \langle \hbar_{\tau_k} - s_{\tau_k}, s - s_{\tau_k} \rangle \leq \langle A\hbar_{\tau_k}, s - s_{\tau_k} \rangle, \forall s \in C$$

This is equivalent to

$$\frac{1}{\lambda_{\tau_k}} \langle \hbar_{\tau_k} - s_{\tau_k}, s - s_{\tau_k} \rangle + \langle A\hbar_{\tau_k}, s_{\tau_k} - \hbar_{\tau_k} \rangle \leq \langle A\hbar_{\tau_k}, s - \hbar_{\tau_k} \rangle, \forall s \in C \tag{23}$$

Next, we show that $\limsup_{k\to\infty} \langle A\hbar_{\tau_k}, s - \hbar_{\tau_k} \rangle \geq 0$. To prove this, we will consider two subcases.

a) If $\limsup_{k\to\infty} \lambda_{\tau_k} > 0$. Since $\{v_{\tau_k}\}$ is bounded, and

$$\lim_{\tau\to\infty} \| v_\tau - \hbar_\tau \| = 0, \lim_{\tau\to\infty} \| v_\tau - s_\tau \| = 0$$

It follows that $\{\hbar_{\tau_k}\}$ and $\{s_{\tau_k}\}$ are bounded. Given that the operator A exhibits uniform continuity over any bounded closed convex subset of H, Lemma 4 allows us to conclude the boundedness of $\{A\hbar_{\tau_k}\}$. Proceeding to the limit as $k \to \infty$ in inequality (23), we get

$$\limsup_{k\to\infty} \langle A\hbar_{\tau_k}, s - \hbar_{\tau_k} \rangle \geq 0$$

b) If $\limsup_{k\to\infty} \lambda_{\tau_k} = 0$. Define

$$t_{\tau_k} = P_C(\hbar_{\tau_k} - \lambda_{\tau_k} l^{-1} A\hbar_{\tau_k})$$

Since $\lambda_{\tau_k} l^{-1} > \lambda_{\tau_k}$, by Lemma 5 we have

$$\| \hbar_{\tau_k} - t_{\tau_k} \| \leq \frac{1}{l} \| \hbar_{\tau_k} - s_{\tau_k} \| \to 0, \text{as } k \to \infty \tag{24}$$

and thus $t_{\tau_k} \rightharpoonup \varpi$. Consequently, $\{t_{\tau_k}\}$ is bounded. Under the assumption that mapping A is uniformly continuous on bounded sets of H, we have

$$\| A\hbar_{\tau_k} - At_{\tau_k} \| \to 0, \text{as } k \to \infty \tag{25}$$

By the line search step size strategy, we know

$$\lambda_{\tau_k} l^{-1} \| AP_C(\hbar_{\tau_k} - \lambda_{\tau_k} l^{-1} A\hbar_{\tau_k}) - A\hbar_{\tau_k} \| > \mu \| \hbar_{\tau_k} - P_C(\hbar_{\tau_k} - \lambda_{\tau_k} l^{-1} A\hbar_{\tau_k}) \|$$

which implies

$$\frac{1}{\mu} \| AP_C(\hbar_{\tau_k} - \lambda_{\tau_k} l^{-1} A\hbar_{\tau_k}) - A\hbar_{\tau_k} \| > \frac{\| \hbar_{\tau_k} - P_C(\hbar_{\tau_k} - \lambda_{\tau_k} l^{-1} A\hbar_{\tau_k}) \|}{\lambda_{\tau_k} l^{-1}} \tag{26}$$

From inequalities (25) and (26), it follows that

$$\lim_{k\to\infty} \frac{\| \hbar_{\tau_k} - P_C(\hbar_{\tau_k} - \lambda_{\tau_k} l^{-1} A\hbar_{\tau_k}) \|}{\lambda_{\tau_k} l^{-1}} = 0 \tag{27}$$

Furthermore, by the definition of t_{τ_k} and Lemma 3, we conclude that

$$\langle \hbar_{\tau_k} - \lambda_{\tau_k} l^{-1} A\hbar_{\tau_k} - t_{\tau_k}, s - t_{\tau_k} \rangle \leq 0, \forall s \in C$$

Therefore,

$$\frac{1}{\lambda_{\tau_k} l^{-1}} \langle \hbar_{\tau_k} - t_{\tau_k}, s - t_{\tau_k} \rangle + \langle A\hbar_{\tau_k}, t_{\tau_k} - \hbar_{\tau_k} \rangle \leq \langle A_{\tau_k}, s - \hbar_{\tau_k} \rangle, \forall s \in C \tag{28}$$

Taking the limit as $k \to \infty$ in inequality (28) and using inequality (24) and Eq.(27), we obtain

$$\limsup_{k\to\infty} \langle A\hbar_{\tau_k}, s - \hbar_{\tau_k} \rangle \geq 0$$

Thus, based on the proofs of subcases a) and b), we conclude that

$$\limsup_{k\to\infty} \langle A\hbar_{\tau_k}, s - \hbar_{\tau_k} \rangle \geq 0 \tag{29}$$

Since

$$\langle As_{\tau_k}, s - s_{\tau_k} \rangle = \langle As_{\tau_k} - A\hbar_{\tau_k}, s - \hbar_{\tau_k} \rangle + \langle A\hbar_{\tau_k}, s - \hbar_{\tau_k} \rangle + \langle As_{\tau_k}, \hbar_{\tau_k} - s_{\tau_k} \rangle \tag{30}$$

and given that $\lim_{k\to\infty} \| \hbar_{\tau_k} - s_{\tau_k} \| = 0$ and the uniform continuity of the mapping A on H, one can conclude that

$$\lim_{k\to\infty} \| A\hbar_{\tau_k} - As_{\tau_k} \| = 0$$

From inequality (29) and Eq.(30), we have

$$\limsup_{k\to\infty} \langle As_{\tau_k}, s - s_{\tau_k} \rangle \geq 0$$

If $\limsup_{k\to\infty} \langle As_{\tau_k}, s - s_{\tau_k} \rangle > 0$, it follows that one can find a subsequence $\{s_{\tau_{k_j}}\}$ of $\{s_{\tau_k}\}$ for which

$$\lim_{k\to\infty} \langle As_{\tau_{k_j}}, s - s_{\tau_{k_j}} \rangle > 0$$

Hence, there exists $N_2 \in \mathbb{N}$ such that

$$\langle As_{\tau_{k_j}}, s - s_{\tau_{k_j}} \rangle > 0, \forall j \geq N_2$$

Since the mapping A is quasimonotone, we have

$$\langle As, s - s_{\tau_{k_j}} \rangle > 0, \forall j \geq N_2$$

Letting $j \to \infty$, we obtain $\varpi \in S_D$.

If $\limsup_{k\to\infty} \langle As_{\tau_k}, s - s_{\tau_k} \rangle = 0$, then

$$\lim_{k\to\infty} \langle As_{\tau_k}, s - s_{\tau_k} \rangle = \lim_{k\to\infty} \sup \langle As_{\tau_k}, s - s_{\tau_k} \rangle = 0$$

We define

$$\vartheta_k = |\langle As_{\tau_k}, s - s_{\tau_k} \rangle| + \frac{1}{k+1}$$

$$\eta_{\tau_k} = \frac{As_{\tau_k}}{\| As_{\tau_k} \|^2}, \forall k \geq N_1$$

Subsequently, we have

$$\langle As_{\tau_k}, s - s_{\tau_k} \rangle + \vartheta_{i_k} > 0, \langle As_{\tau_k}, \eta_{\tau_k} \rangle = 1$$

Furthermore, we have

$$\langle As_{\tau_k}, s + \vartheta_k \eta_{\tau_k} - s_{\tau_k} \rangle > 0, \forall k \geq N_1$$

Since the mapping A is quasimonotone, it follows that

$$\langle A(s + \vartheta_k \eta_{\tau_k}), s + \vartheta_k \eta_{\tau_k} - s_{\tau_k} \rangle > 0, \forall k \geq N_1$$

For $k \geq N_1$, we have

$$\langle As, s - s_{\tau_k} \rangle = \langle As - A(s + \vartheta_k \eta_{\tau_k}), s + \vartheta_k \eta_{\tau_k} - s_{\tau_k} \rangle + \langle A(s + \vartheta_k \eta_{\tau_k}), s + \vartheta_k \eta_{\tau_k} - s_{\tau_k} \rangle - \langle As,$$

$$\vartheta_k\eta_{\tau_k}\rangle \geq \langle As - A(s + \vartheta_k\eta_{\tau_k}), s + \vartheta_k\eta_{\tau_k} - s_{\tau_k}\rangle - \langle As, \vartheta_k\eta_{\tau_k}\rangle \quad (31)$$

Next, we show that $\lim_{k\to\infty}\vartheta_k\eta_{\tau_k} = 0$. Since $\|\eta_{\tau_k}\| = \frac{1}{\|As_{\tau_k}\|} < \frac{2}{d}, \forall k \geq N_1$ is bounded and $\lim_{k\to\infty}\vartheta_k = 0$, it follows that $\lim_{k\to\infty}\vartheta_k\eta_{\tau_k} = 0$. Taking the limit as $k \to \infty$, since the function A exhibits uniform continuity, the sequence $\{s_{\tau_k}\}$ is bounded, and $\lim_{k\to\infty}\vartheta_k\eta_{\tau_k} = 0$, the expression on the right-hand side of inequality (31) vanishes in the limit. Therefore, we obtain

$$\langle As, s - \varpi\rangle \geq 0, \forall s \in C$$

which implies $\varpi \in S_D$.

Lemma 10 Under the conditions (A)-(E), it can be established that the sequence $\{\upsilon_\tau\}$ yielded by Algorithm 1 exhibits a finite collection of weak accumulation points, with each such point being an element of the solution set S.

Proof We first prove that the sequence $\{\upsilon_\tau\}$ possesses no more than a single weak cluster point in S_D. Assume, to the contrary, that $\{\upsilon_\tau\}$ contains two or more distinct weak accumulation points $\varpi^1 \in S_D$ and $\varpi^2 \in S_D$ with $\varpi^1 \neq \varpi^2$. Then there exist two subsequences $\{\upsilon_{\tau_k}\}$ and $\{\upsilon_{l_k}\}$ of $\{\upsilon_\tau\}$ such that

$$\upsilon_{\tau_k} \rightharpoonup \varpi^1 \text{ and } \upsilon_{l_k} \rightharpoonup \varpi^2, \text{as } k \to \infty$$

Furthermore, by Lemma 2 and Lemma 7, the limit $\lim_{\tau\to\infty}\|\upsilon_\tau - \varpi\|$ exists for any $\varpi \in S_D$, we have

$$\lim_{\tau\to\infty}\|\upsilon_\tau - \varpi^1\| = \lim_{k\to\infty}\|\upsilon_{\tau_k} - \varpi^1\| = \liminf_{k\to\infty}\|\upsilon_{\tau_k} - \varpi^1\| < \liminf_{k\to\infty}\|\upsilon_{\tau_k} - \varpi^2\| = \lim_{\tau\to\infty}\|\upsilon_\tau - \varpi^2\| = \lim_{k\to\infty}\|\upsilon_{l_k} - \varpi^2\| = \liminf_{k\to\infty}\|\upsilon_{l_k} - \varpi^2\| < \liminf_{k\to\infty}\|\upsilon_{l_k} - \varpi^1\| = \lim_{k\to\infty}\|\upsilon_{l_k} - \varpi^1\| = \lim_{\tau\to\infty}\|\upsilon_\tau - \varpi^1\|$$

This results in a contradiction. Consequently, $\{\upsilon_\tau\}$ admits no more than a single weak cluster point in S_D. Moreover, according to $T = \{u \in C: A(u) = 0\} \backslash S_D$ is a finite set and Lemma 9, we deduce that $\{\upsilon_\tau\}$ has finitely many weak cluster points in S.

Lemma 11 Suppose assumptions (A)-(E) hold and the sequence $\{\upsilon_\tau\}$ has finitely many weak cluster points $\varpi^1, \varpi^2, \cdots \varpi^m$. From this, there is at least one positive integer N for which, for all $\tau \geq N$, $\upsilon_\tau \in \Theta$, where

$$\Theta = \bigcup_{t=1}^{m}\Theta^t, \quad \Theta^d = \bigcap_{t=1, t\neq d}^{m}\{u: \langle u, \frac{\varpi^d - \varpi^t}{\|\varpi^d - \varpi^t\|}\rangle > \gamma_0 + \frac{\|\varpi^d\|^2 - \|\varpi^t\|^2}{2\|\varpi^d - \varpi^t\|}\}$$

and

$$\gamma_0 = \min\{\frac{\|\varpi^d - \varpi^t\|}{4}: t, d \in \{1, 2, \cdots, m\}, t \neq d\}$$

Proof Let $\{\upsilon_{\tau_k}^d\}$ be a subsequence of $\{\upsilon_\tau\}$ such that $\upsilon_{\tau_k}^d \rightharpoonup \varpi^d$ as $k \to \infty$ .Then we know that

$$\lim_{k\to\infty}\langle\upsilon_{\tau_k}^d, \varpi^d - \varpi^t\rangle = \langle\varpi^d, \varpi^d - \varpi^t\rangle, \forall t \neq d$$

For $t \neq d$, we have

$$\langle\varpi^d, \varpi^d - \varpi^t\rangle = \|\varpi^d\|^2 - \langle\varpi^d, \varpi^t\rangle = \frac{\|\varpi^d - \varpi^t\|^2}{2} + \frac{\|\varpi^d\|^2}{2} - \frac{\|\varpi^t\|^2}{2} > \frac{\|\varpi^d - \varpi^t\|^2}{4} + \frac{\|\varpi^d\|^2}{2} - \frac{\|\varpi^t\|^2}{2} \quad (32)$$

Therefore, for all $\forall t \neq d$,

$$\langle\varpi^d, \frac{\varpi^d - \varpi^t}{\|\varpi^d - \varpi^t\|}\rangle > \frac{\|\varpi^d - \varpi^t\|}{4} + \frac{\|\varpi^d\|^2 - \|\varpi^t\|^2}{2\|\varpi^d - \varpi^t\|} \quad (33)$$

From inequalities (32) and (33), when k is sufficiently large, we have

$$\upsilon_{\tau_k}^d \in \{u: \langle u, \frac{\varpi^d - \varpi^t}{\|\varpi^d - \varpi^t\|}\rangle > \frac{\|\varpi^d - \varpi^t\|}{4} + \frac{\|\varpi^d\|^2 - \|\varpi^t\|^2}{2\|\varpi^d - \varpi^t\|}\}$$

Hence, for sufficiently large k,

$$\upsilon_{\tau_k}^d \in \Theta^d$$

where

$$\Theta^d = \bigcap_{t=1, t\neq d}^{m}\{u: \langle u, \frac{\varpi^d - \varpi^t}{\|\varpi^d - \varpi^t\|}\rangle > \gamma_0 + \frac{\|\varpi^d\|^2 - \|\varpi^t\|^2}{2\|\varpi^d - \varpi^t\|}\} \quad (34)$$

and

$$\gamma_0 = \min\left\{\frac{\|\varpi^d - \varpi^t\|}{4}: t, d \in \{1, 2, \cdots, m\}, t \neq d\right\}$$

From the above, it necessarily follows that

$$\Theta^d = \bigcap_{t=1, t\neq d}^{m}\{u: \langle -u, \frac{\varpi^d - \varpi^t}{\|\varpi^d - \varpi^t\|}\rangle < -\gamma_0 + \frac{\|\varpi^t\|^2 - \|\varpi^d\|^2}{2\|\varpi^d - \varpi^t\|}\} \quad (35)$$

We define $\Theta = \bigcup_{t=1}^{m}\Theta^t$. It will be shown that $\upsilon_\tau \in \Theta$ whenever τ is sufficiently large. Assume that one can find a subsequence $\{\upsilon_{\tau_k}\}$ of $\{\upsilon_\tau\}$ satisfying the property that $\upsilon_{\tau_k} \notin \Theta$ holds for all positive integers k. Given the boundedness of $\{\upsilon_{\tau_k}\}$, it follows that a subsequence of $\{\upsilon_{\tau_k}\}$ converges weakly to $\bar{\varpi} \in C$. For simplicity, let us continue to denote this subsequence as $\{\upsilon_{\tau_k}\}$ with $\upsilon_{\tau_k} \rightharpoonup \bar{\varpi}$. Because $\upsilon_{\tau_k} \notin \Theta$, for every $d \in \{1, 2, \cdots, m\}$,

$$v_{\tau_k} \notin \Theta^d = \bigcap_{t=1,t\neq d}^{m} \{u:\langle u,\frac{\varpi^d - \varpi^t}{\|\varpi^d - \varpi^t\|}\rangle > \gamma_0 + \frac{\|\varpi^d\|^2 - \|\varpi^t\|^2}{2\|\varpi^d - \varpi^t\|}\}$$

Applying the pigeonhole principle, one can find a subsequence $\{v_{\tau_{k_l}}\}$ of $\{v_{\tau_k}\}$ and an index $d_0 \in \{1,2,\cdots,m\}\setminus\{d\}$ such that for all $l \geqslant 0$,

$$v_{\tau_{k_l}} \notin \{u:\langle u,\frac{\varpi^d - \varpi^{d_0}}{\|\varpi^d - \varpi^{d_0}\|}\rangle > \gamma_0 + \frac{\|\varpi^d\|^2 - \|\varpi^{d_0}\|^2}{2\|\varpi^d - \varpi^{d_0}\|}\}$$

and consequently,

$$\bar{\varpi} \notin \{u:\langle u,\frac{\varpi^d - \varpi^{d_0}}{\|\varpi^d - \varpi^{d_0}\|}\rangle > \gamma_0 + \frac{\|\varpi^d\|^2 - \|\varpi^{d_0}\|^2}{2\|\varpi^d - \varpi^{d_0}\|}\} \tag{36}$$

From inequalities (33), (36) and the inequality

$$\langle \varpi^d,\frac{\varpi^d - \varpi^{d_0}}{\|\varpi^d - \varpi^{d_0}\|}\rangle > \frac{\|\varpi^d - \varpi^{d_0}\|}{4} + \frac{\|\varpi^d\|^2 - \|\varpi^{d_0}\|^2}{2\|\varpi^d - \varpi^{d_0}\|} \geqslant \gamma_0 + \frac{\|\varpi^d\|^2 - \|\varpi^{d_0}\|^2}{2\|\varpi^d - \varpi^{d_0}\|}$$

we conclude that $\bar{\varpi} \neq \varpi^d$. Due to the arbitrariness of d, we have $\bar{\varpi} \notin \{\varpi^1,\varpi^2,\cdots,\varpi^m\}$, which cannot exist. Therefore, $v_\tau \in \Theta$ for all sufficiently large τ. Accordingly, one may select a positive integer N such that the property $v_\tau \in \Theta$ is satisfied for every $\tau \geqslant N$.

Theorem 1 Assuming that assumptions (A)-(E) are met, the sequence $\{v_\tau\}$ generated by Algorithm 1 converges weakly to $\varpi \in S$.

Proof From Lemma 7, we have $\lim_{\tau\to\infty}\|v_{\tau+1} - v_\tau\| = 0$. Therefore, one can find $N' > N$ such that for every $\tau \geqslant N'$, $\|v_{\tau+1} - v_\tau\| < \gamma_0$. Suppose $\{v_\tau\}$ has two or more weak cluster points. Then by Lemma 11, there exists $N'' \geqslant N' > N$ such that $v_{N''+1} \in \Theta^t$ and $v_{N''} \in \Theta^d$, where $t \neq d$ and $t,d \in \{1,2,\cdots,m(m \geqslant 2)\}$. Specifically, we have $\|v_{N''+1} - v_{N''}\| < \gamma_0$. However, from Eqs. (34) and (35), we have

$$v_{N''} \in \Theta^d = \bigcap_{t=1,t\neq d}^{m} \left\{u:\langle u,\frac{\varpi^d - \varpi^t}{\|\varpi^d - \varpi^t\|}\rangle > \gamma_0 + \frac{\|\varpi^d\|^2 - \|\varpi^t\|^2}{2\|\varpi^d - \varpi^t\|}\right\}$$

and

$$v_{N''+1} \in \Theta^t = \bigcap_{d=1,d\neq t}^{m} \left\{u:\langle -u,\frac{\varpi^t - \varpi^d}{\|\varpi^t - \varpi^d\|}\rangle < -\gamma_0 + \frac{\|\varpi^d\|^2 - \|\varpi^t\|^2}{2\|\varpi^t - \varpi^d\|}\right\}$$

Furthermore, we have

$$\langle v_{N''},\frac{\varpi^d - \varpi^t}{\|\varpi^d - \varpi^t\|}\rangle > \gamma_0 + \frac{\|\varpi^d\|^2 - \|\varpi^t\|^2}{2\|\varpi^d - \varpi^t\|} \tag{37}$$

and

$$\langle -v_{N''+1},\frac{\varpi^d - \varpi^t}{\|\varpi^t - \varpi^d\|}\rangle > \gamma_0 + \frac{\|\varpi^t\|^2 - \|\varpi^d\|^2}{2\|\varpi^t - \varpi^d\|} \tag{38}$$

Combining inequalities (37) and (38), we get

$$2\gamma_0 < \langle v_{N''} - v_{N''+1},\frac{\varpi^d - \varpi^t}{\|\varpi^t - \varpi^d\|}\rangle \leqslant \|v_{N''} - v_{N''+1}\| < \gamma_0$$

which is impossible. Therefore, within the set of solution S, the sequence $\{v_\tau\}$ converges weakly to a single point, i.e., $v_\tau \rightharpoonup \varpi$.

3 Numerical Experiment

To illustrate the capabilities of Algorithm 1, we carry out a series of numerical tests in this section. All experimental results were obtained using MATLAB R2023b on a PC equipped with an Intel (R) Core (TM) i7-6700 processor (base frequency 3.40 GHz, max turbo frequency 3.41 GHz) and 16.00 GB of RAM. For the current procedure, the algorithm halts when the following condition $\|v_{\tau+1} - v_\tau\| \leqslant \varepsilon$ is satisfied. Here, Iter stands for the total number of iterations and T represents the CPU time consumed by the algorithm.

Example 1 The feasible set Ω is defined as the standard unit ball:

$$\Omega = \{t = (t_1,t_2)^{\mathrm{T}} \in \mathbb{R}^2 \mid \|t\|_2 \leqslant 1\}$$

Define the mapping $G:\Omega \to \mathbb{R}^2$ as follows:

$$G(t) = \begin{pmatrix} t_1 + t_2 + e^{t_1} \\ -t_1 + t_2 + e^{t_2} \end{pmatrix}$$

It can be verified that G is a monotone mapping. Table 1 presents a comparison of our Algorithm 1 with Algorithm 3.1 from Ref. [15] and Algorithm 1 from Ref. [19] under different error tolerances. As shown in Figs. 1 and 2, our algorithm requires significantly fewer iterations to converge compared with the other two benchmark algorithms. The initial point is set to $v_0 = (1,2)$, $v_1 = (0.50,0.75)$ and the parameter values for both algorithms are specified in Table 2.

Table 1 Computation time and iteration steps under different error tolerances

Error ε	Algorithm 1		Algorithm 3.1 in Ref.[15]		Algorithm 1 in Ref.[19]	
	T(s)	Iter	T(s)	Iter	T(s)	Iter
10^{-4}	3.89×10^{-4}	39	7.1×10^{-3}	142	3.9×10^{-3}	47
10^{-6}	5.90×10^{-4}	58	8.9×10^{-3}	1424	4.5×10^{-3}	72

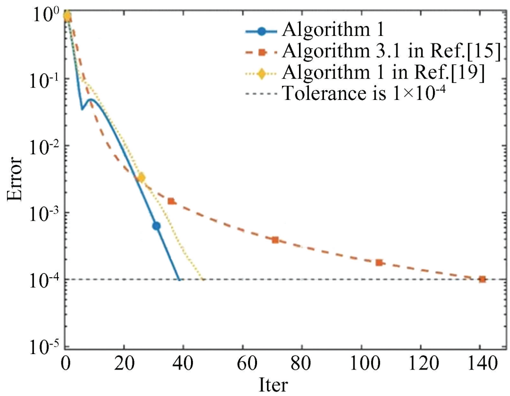

Fig.1 $\varepsilon = 10^{-4}$

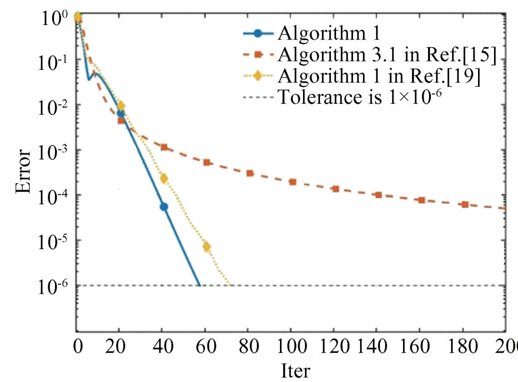

Fig.2 $\varepsilon = 10^{-6}$

Table 2 Parameter settings for the algorithms

Algorithm 1	Algorithm 3.1 in Ref.[15]	Algorithm 1 in Ref.[19]
$\sigma = 0.495$, $\theta_\tau = 1$, $\xi_\tau = 0.225$, $\gamma = 0.5$, $l = 0.5$, $\mu = 0.99$, $\lambda_1 = 1.1$	$\gamma = 0.5$, $l = 0.5$, $\mu = 0.99$, $\lambda_1 = 1.1$, $\beta_\tau = \frac{1}{\tau+2}$, $f(x) = \frac{1}{8}x$	$\delta = 0.495$, $\theta_\tau = 1$, $\alpha_\tau = 0.225$, $\mu = 0.99$, $\lambda_1 = 1.1$

Example 2 The Hilbert space under consideration consists of all sequences with finite squared sum, denoted $H = l^2$. Define the set

$$C := \{p = (p_1, p_2, \cdots, p_\tau, \cdots) \in H : |p_\tau| \leqslant \frac{1}{\tau}, \tau = 1, 2, \cdots, n, \cdots\}$$

Consider the operator $A: C \to H$ defined through

$$At = \left(\|t\| + \frac{1}{\|t\| + \alpha}\right) t$$

where $\alpha > 0$. As established in Ref.[26], the problem has a unique solution $S = \{0\}$. On H, the mapping A satisfies pseudomonotone behavior, while it maintains uniform continuity and weak sequential continuity on the subset C. Nevertheless, it does not exhibit Lipschitz behavior over H. For the experiments presented below, we assign the following values $\alpha = 0.5$ and $H = \mathbb{R}^m$ (with m taking different values). For this scenario, C represents the set of feasible points given by:

$$C = \{p \in \mathbb{R}^m : -\frac{1}{\tau} \leqslant p_\tau \leqslant \frac{1}{\tau}, \tau = 1, 2, \cdots, m\}$$

In this instance, we determine that the algorithm terminates when the following condition is satisfied $\|v_{\tau+1} - v_\tau\| \leqslant 10^{-6}$. A comparison is made between our algorithm and Algorithm 2 in Ref.[27], as well as Algorithm 3.1 in Ref.[15]. A graphical and tabular summary of the results is provided in Table 3, Figs.3 and 4. The algorithm achieves the best performance, considering both how many iterations are required and the time needed for computation. The parameters used are chosen as follows:

Algorithm 1: $\sigma = 0.495, \theta_\tau = 1, \xi_\tau = 0.225, \gamma = 0.3, l = 0.5, \mu = 0.8, \lambda_1 = 0.1$.

Algorithm 2 in Ref.[27]: $\alpha_\tau = \frac{1}{\tau^{0.5}}, \gamma = 0.5, l = 0.3, \mu = 0.8, \beta_\tau = \frac{1}{\tau+2}, f(x) = \frac{1}{8}x, \lambda_1 = 0.1$.

Algorithm 3.1 in Ref.[15]: $\gamma = 0.3, l = 0.5, \beta_\tau = \frac{1}{i+2}, f(x) = \frac{1}{8}x, \mu = 0.8, \lambda_1 = 0.1$.

Table 3 Performance comparison of all algorithms for different *m*

m	Algorithm 1		Algorithm 2 in Ref.[27]		Algorithm 3.1 in Ref.[15]	
	Iter	T(s)	Iter	T(s)	Iter	T(s)
150	123	0.0015	168	0.0049	190	0.0081
500	193	0.0059	276	0.0068	210	0.0091

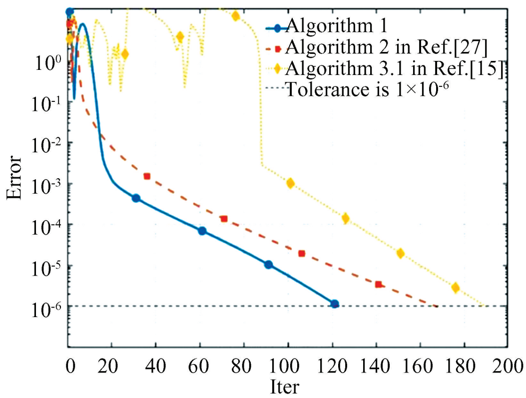

Fig.3 $m = 150$

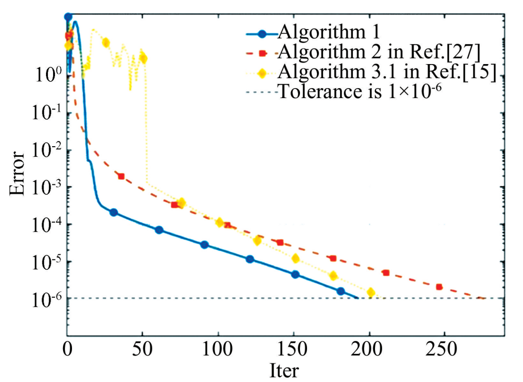

Fig.4 $m = 500$

Example 3 Let $H = L^2([0,1])$ be accompanied by the inner product defined via integration $\langle p,q\rangle = \int_0^1 p(t)q(t)\mathrm{d}t$ along with its induced norm $\|p\| = \langle p,p\rangle^{\frac{1}{2}}$. Consider the constraint set $C = \left\{p \in L^2([0,1]) : \int_0^1 tp(t)\mathrm{d}t = 2\right\}$. An operator F is constructed via

$$Fp(t) := \max\{p(t),0\},p \in L^2([0,1]),t \in [0,1]$$

This mapping F satisfies monotonicity and global Lipschitz continuity, where the corresponding Lipschitz bound is unity. The projection of an element $p \in L^2([0,1])$ onto C is given explicitly by

$$P_C p(t) = p(t) - \frac{\int_0^1 tp(t)\,\mathrm{d}t}{\int_0^1 t^2\mathrm{d}t}t,t \in [0,1]$$

We choose the initial points as $v_0(t) = \frac{1}{13}(97t^2 + 4t)$ and $v_1(t) = \frac{1}{250}(t^2 - \mathrm{e}^{-7t})$. Our algorithm will be compared with Algorithm 1 in Ref.[19] and Algorithm 3.1 in Ref.[28]. Figs.5 and 6 display the outcomes obtained in the last stage of the analysis. It can be observed that, under the tolerances 10^{-4} and 10^{-6}, our algorithm requires fewer iterations than the other methods. The following provides the parameter configurations for every algorithm:

Algorithm 1: $\sigma = 0.495,\theta_\tau = 1,\xi_\tau = 0.225,\gamma = 0.7,l = 0.5,\mu = 0.99,\lambda_1 = 1.1$.

Algorithm 1 in Ref.[19]: $\delta = 0.495,\theta_\tau = 1,\mu = 0.99,\alpha_\tau = 0.225,\lambda_1 = 1.1$.

Algorithm 3.1 in Ref.[28]: $\alpha_\tau = 0.225,\alpha = 2,\mu = 0.99,\lambda_1 = 1.1$.

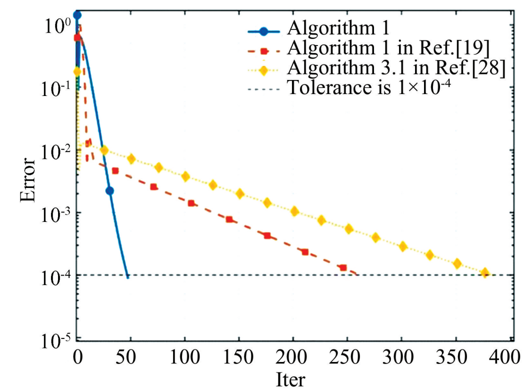

Fig.5 $\varepsilon = 10^{-4}$

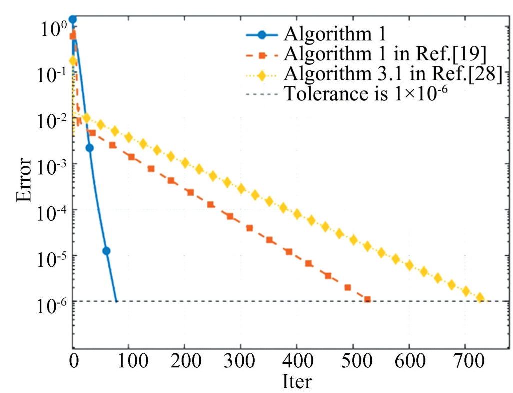

Fig.6 $\varepsilon = 10^{-6}$

4 Conclusions

A relaxed iterative scheme enhanced by a double inertial mechanism and an adaptive line-search strategy is developed in this work to handle quasimonotone and uniformly continuous VIP in real Hilbert spaces. Unlike the usual assumptions in the study of quasimonotone variational inequalities, where the mapping A is required to satisfy $Ax \neq 0, \forall x \in H$, the quasimonotone variational inequality problem we consider encompasses this case. Under appropriate conditions, this work demonstrates the weak convergence of the algorithmically generated sequence $\{v_\tau\}$ towards a solution of VIP. To further demonstrate its effectiveness, computational tests are carried out to assess the practical performance of the developed approach.

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