Journal of Harbin Institute of Technology (New Series)

Citation: Anjali, Man Singh Beniwal, Mohd Atif Wahid. Several new integral and discrete operators based on special function. *Journal of Harbin Institute of Technology(New Series)*.DOI:10.11916/j.issn.1005-9113.2025095

Several New Integral and Discrete Operators Based on Special Function

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Abstract: This study uses the classical orthogonal Laguerre polynomials as a foundation for constructing new operators. We present the Laguerre-Păltănea operator based on the modified Laguerre polynomial that facilitates the approximation of integrable functions. Furthermore, some new discrete and integral-type operators derived from the combination of the Laguerre operator, Laguerre-Păltănea operator, and Szász-Mirakyan operator are proposed. We compare the error values of the operators and emphasize which operator provides the best approximation. Ultimately, Mathematica software is utilized to generate graphs and infer the convergence of the stated operators to a function for various parameter values under consideration. Additionally, we compare each specified operator and observe that the proposed Laguerre-Păltănea operator under discussion offers a better approximation than the other potential compositions.

Keywords: Laguerre polynomial; convergence properties; Păltănea operators; error's numerical interpretation; geometric comparison

CLC number: O174 **Document code**: A **Article ID**: 1005-9113(2025)00-0000-09

0 Introduction

Approximation theory is a systematic theoretical study of techniques that employ numerical approaches to solve various problems of mathematical analysis. In practice, it typically refers to the application of computer simulations and other computational methods in various scientific fields. Approximation theory, especially estimation by polynomials and piecewise polynomials, has long been closely associated with Computer-Aided Geometric Design (CAGD); where the design, calculation, and computer representation of curved shapes are the main focus of CAGD. Approximation theory is widely used in machine learning, as several machine learning algorithms rely on approximating complex functions to establish relationships between input data and output labels.

Optimization algorithms often employ approximation approaches to explore the search space and identify suitable solutions effectively. It is also an essential part of modern artificial intelligence, as it approximates complex functions by composing simpler ones using layers of interconnected neurons.

In this study, we compose various operators to define new discrete and integral type operators based on a special function, Laguerre polynomials. These polynomials are the most significant classical orthogonal polynomials. They have applications in a variety of mathematical disciplines. Additionally, these polynomials have provided numerous applications in physics, engineering, and other fields, including plasma physics, quantum mechanics, and the measurement of symmetric and central vortex states in Bose-Einstein condensates in 3-D with cylindrical symmetry[1]. Further, the computation of weights and nodes in the Gauss-Laguerre quadrature strongly depends on the evaluation of Laguerre polynomials. Here, we mention some research on generating functions, closed forms, and efficient computation of Laguerre polynomials (see Refs.[2-4]). Recently, Gupta [5] explored a discrete operator based on the modified Laguerre polynomial.

We start by recalling the well-known special

Received 2025-07-31.

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function, Laguerre polynomials $L_k(x)$, which are solutions to the Laguerre differential equations given by:

$$x\,D^2 y + (1 - x)Dy + ky = 0 \qquad (1)$$

where D^n represents n -th derivative of y with respect to x , and Eq. (1) is a second-order linear differential equation that has non-singular solutions only if k is a non-negative integer. The closed form for kth Laguerre polynomial and the generating function, as mentioned in Ref. [2], have the following forms, respectively:

$$L_k(x) = \sum_{r=0}^{k} \binom{k}{r} \frac{(-1)^r}{r!} x^r$$

and

$$\sum_{k=0}^{\infty} y^k L_k(x) = e^{\frac{-xy}{1-y}} \cdot \frac{1}{1-y}$$

Now, considering the Generalized Laguerre polynomial $L_k^\alpha(x)$, for $\alpha > -1$, which is defined as the following differential equation's solution:

$$x\,D^2 y + (\alpha + 1 - x)Dy + ky = 0$$

The closed form of the k^{th} Generalized Laguerre polynomial and the generating function, as mentioned in Ref.[4], have the following forms, respectively:

$$L_k^\alpha(x) = \sum_{r=0}^{k} \binom{\alpha + k}{\alpha + r} \frac{(-1)^r}{r!} x^r$$

and

$$\sum_{k=0}^{\infty} y^k L_k^\alpha(x) = e^{\frac{-xy}{1-y}} \cdot \frac{1}{(1-y)^{\alpha+1}} \qquad (2)$$

Remark 1 For $x \in (-\infty, 0]$ and $\alpha > -1$, we have $L_k^\alpha(x) > 0$.

In particular,

$$L_0^\alpha(x) = 1$$

$$L_1^\alpha(x) = 1 + \alpha - x$$

Following the modified generating function for Generalized Laguerre polynomials, as given in Eq.(3. 3) of Ref.[4], we define

$$\sum_{k=0}^{\infty} L_k^\alpha\left(-\frac{x}{2}\right) y^k = \exp\left(\frac{xy}{2(1-y)}\right) \cdot \frac{1}{(1-y)^{\alpha+1}}, \ |y| < 1 \qquad (3)$$

Sucu et al.[4] proposed a positive linear operator for $x \in [0, \infty)$ as:

$$(W_n^\alpha f)(x) = \sum_{k=0}^{\infty} l_{n,k}^\alpha(x) f\left(\frac{k}{n}\right) \qquad (4)$$

where

$$l_{n,k}^\alpha(x) := e^{\frac{-nx}{2}} \cdot 2^{-\alpha-k-1} \cdot L_k^\alpha\left(\frac{-nx}{2}\right)$$

with $\alpha > -1, n \in \mathbb{N}, f\colon [0, \infty) \to \mathbb{R}$ is a continuous function and $L_k^\alpha(-x)$ is the modified Laguerre polynomials defined by means of confluent hypergeometric series as follows:

$$L_k^\alpha(-x) := \frac{(\alpha+1)_k}{k!}\,{}_1F_1(-k; \alpha+1; -x) = \binom{k+\alpha}{\alpha}\,{}_1F_1(-k; \alpha+1; -x)$$

We denote $\exp_A(y) = e^{Ay}$.

Remark 2 Taking into account Eq. (2), we obtain

$$(W_n^\alpha \exp_A)(x) = \exp\left(\frac{nx(e^{\frac{A}{n}} - 1)}{2 - e^{\frac{A}{n}}}\right) \cdot \frac{1}{(2 - e^{\frac{A}{n}})^{1+\alpha}}$$

1 New Integral Operator based on Laguerre Polynomials

In many mathematical theories, estimating a function using positive linear operators is a significant concern. Summation-integral type operators have been the subject of several published works[6-15]. Here, we consider the Păltănea variant of the operator based on the well-known special function, Laguerre polynomials. Various engineering problems have unbounded domains, such as fluid dynamics, signal processing, heat transfer, and control theory. The Laguerre Păltănea operators are very useful to address these problems. These operators act as a bridge between practical modeling requirements and theoretical approximation concepts. To solve differential equations, optimize system responses, and enhance computational algorithms, these operators have proven to be an interesting mathematical foundation. Thus, they highlight the increasing importance of this synthesis of pure mathematics and applied mathematics in modern engineering science. Some examples of this connection can be found in Refs. [16 - 19]. Consider the Laguerre-Păltănea operators for $\rho > 0, \alpha > -1$ and for $x \in [0, \infty)$ given by

$$(L_{n,\rho}^\alpha f)(x) = \sum_{k=0}^{\infty} l_{n,k}^\alpha(x)\, A_{n,k}^\rho(f) \qquad (5)$$

where

$$A_{n,k}^\rho(f) = \begin{cases} \left\langle \dfrac{n\rho}{\Gamma(k\rho)} \cdot e^{-n\rho y} \cdot (n\rho y)^{k\rho - 1}, f \right\rangle, & 1 \leqslant k < \infty \\ f(0), & k = 0 \end{cases}$$

with the inner product provided by $\langle f,g\rangle = \int\limits_0^\infty f(s)g(s)\,\mathrm{d}s$, *and* $f: [0,\infty) \to \mathbb{R}$ is an integrable function for which the above integral and series converge. We can also consider the integral representation of Eq.(5) given by:

$$(L_{n,\rho}^{\alpha}f)(x) = \int\limits_0^\infty K_{n,x}^{\rho,\alpha}(y)f(y)\,\mathrm{d}y$$

where

$$K_{n,x}^{\rho,\alpha}(y) = \sum_{k=1}^{\infty} l_{n,k}^{\alpha}(x)\cdot s_{n,k}^{\rho}(y) + l_{n,0}^{\alpha}(x)\cdot\delta(y)$$

$$s_{n,k}^{\rho}(y) = \frac{(n\rho)^{k\rho}}{\Gamma(k\rho)}\cdot \mathrm{e}^{-n\rho y}\cdot y^{k\rho-1}$$

and $\delta(y)$ is the Dirac delta function, the so-called inverse Laplace transform of 1.

Proposition 1 The function that generates the moments for the operator $L_{n,\rho}^{\alpha}$ is given by

$$(L_{n,\rho}^{\alpha}\exp_A)(x) = \exp\left[-nx\left(\frac{(n\rho-A)^{\rho}-(n\rho)^{\rho}}{2(n\rho-A)^{\rho}-(n\rho)^{\rho}}\right)\right]\cdot \left[\frac{(n\rho-A)^{\rho}}{2(n\rho-A)^{\rho}-(n\rho)^{\rho}}\right]^{1+\alpha}$$

Proof Consider

$$(L_{n,\rho}^{\alpha}\exp_A)(x) = \sum_{k=0}^{\infty} l_{n,k}^{\alpha}(x)\cdot A_{n,k}^{\rho}(\exp_A) = \sum_{k=1}^{\infty} l_{n,k}^{\alpha}(x)\frac{(n\rho)^{k\rho}}{\Gamma(k\rho)}\int_0^\infty \mathrm{e}^{-(n\rho-A)y}\cdot y^{k\rho-1}\mathrm{d}y + l_{n,0}^{\alpha}(x) = \sum_{k=0}^{\infty} l_{n,k}^{\alpha}(x)\left(\frac{n\rho}{n\rho-A}\right)^{k\rho}$$

Utilizing Eq.(3), we have

$$(L_{n,\rho}^{\alpha}\exp_A)(x) = \mathrm{e}^{-\frac{nx}{2}}\cdot\exp\left(\frac{nx(n\rho)^{\rho}}{2(2(n\rho-A)^{\rho}-(n\rho)^{\rho})}\right)\cdot \left[\frac{(n\rho-A)^{\rho}}{2(n\rho-A)^{\rho}-(n\rho)^{\rho}}\right]^{1+\alpha}$$

On simplifying further, we get the required result.

Lemma 1 Using Proposition 1, we conclude that for certain constants b_i and $e_i(t) = t^i, i = 0,1,2,\cdots, b_i \neq 0$, we have

$$\left(L_{n,\rho}^{\alpha}\sum_{i\geqslant 0} b_i\, e_i\right)(x) = b_0 + b_1\left[x + \frac{1+\alpha}{n}\right] + b_2\left[x^2 + \frac{x}{\rho n^2}(2\alpha n\rho + 5n\rho + n) + \frac{1}{\rho n^2}(\alpha^2\rho + 4\alpha\rho + 3\rho + \alpha + 1)\right] + \cdots$$

Further, for some constants $c_i, i = 0,1,2,\cdots, c_i \neq 0$, we have

$$\left(L_{n,\rho}^{\alpha}\sum_{i\geqslant 0} c_i\,(e_1 - x\,e_0)^i\right)(x) = c_0 + c_1\left[\frac{1+\alpha}{n}\right] + c_2\left[\frac{x}{\rho n}(3\rho+1) + \frac{1}{\rho n^2}(\alpha^2\rho + 4\alpha\rho + 3\rho + \alpha + 1)\right] + \cdots\cdots$$

Denote

$$\psi_{n,\rho}^{\alpha}(x) := \frac{x}{\rho n}(3\rho+1) + \frac{1}{\rho n^2}\left(\alpha^2\rho + 4\alpha\rho + 3\rho + \alpha + 1\right)$$

Theorem 1 For $f \in C_{BD}(\mathbb{R}^+)$ (the class of all bounded functions on the non-negative real axis which are continuous), $r \geqslant 1, s \in \mathbb{R}$ and $x \in [0,\infty)$, the subsequent limits hold true:

1) $\lim\limits_{n\to\infty}(L_{rn,\rho}^{\alpha}f(ny))\left(\frac{x}{n}\right) = (L_{r,\rho}^{\alpha}f(y))(x)$

2) $\lim\limits_{n\to\infty}\left(L_{r,\rho}^{\alpha}f\left(\frac{y}{n}\right)\right)(nx) = f(x)$

3) $\lim\limits_{\rho\to\infty}(L_{n,\rho}^{\alpha}f)(x) = (W_n^{\alpha}f)(x)$

Proof Using Proposition 1, with i denoting the imaginary part consider

$$\lim_{n\to\infty}(L_{rn,\rho}^{\alpha}\,\mathrm{e}^{\mathrm{i}sny})\left(\frac{x}{n}\right) = \lim_{n\to\infty}\exp\left[-rx\left(\frac{(nr\rho-\mathrm{i}sn)^{\rho}-(nr\rho)^{\rho}}{2(nr\rho-\mathrm{i}sn)^{\rho}-(nr\rho)^{\rho}}\right)\right]\cdot\left(\frac{(nr\rho-\mathrm{i}sn)^{\rho}}{2(nr\rho-\mathrm{i}sn)^{\rho}-(nr\rho)^{\rho}}\right)^{1+\alpha} = \exp\left[-rx\left(\frac{(r\rho-\mathrm{i}s)^{\rho}-(r\rho)^{\rho}}{2(r\rho-\mathrm{i}s)^{\rho}-(r\rho)^{\rho}}\right)\right]\cdot\left[\frac{(r\rho-\mathrm{i}s)^{\rho}}{2(r\rho-\mathrm{i}s)^{\rho}-(r\rho)^{\rho}}\right]^{1+\alpha} = (L_{r,\rho}^{\alpha}\,\mathrm{e}^{\mathrm{i}sy})(x)$$

Next

$$\lim_{n\to\infty}(L_{r,\rho}^{\alpha}\,\mathrm{e}^{\frac{\mathrm{i}sy}{n}})(nx) = \lim_{n\to\infty}\exp\left[-rnx\left(\frac{\left(r\rho-\frac{\mathrm{i}s}{n}\right)^{\rho}-(r\rho)^{\rho}}{2\left(r\rho-\frac{\mathrm{i}s}{n}\right)^{\rho}-(r\rho)^{\rho}}\right)\right]\cdot\left(\frac{\left(r\rho-\frac{\mathrm{i}s}{n}\right)^{\rho}}{2\left(r\rho-\frac{\mathrm{i}s}{n}\right)^{\rho}-(r\rho)^{\rho}}\right)^{1+\alpha} = \mathrm{e}^{\mathrm{i}sx} = \mathrm{Id}(\mathrm{e}^{\mathrm{i}sy};x)$$

where Id denotes the identity operator. Finally

$$\lim_{\rho\to\infty}(L_{n,\rho}^{\alpha}\,\mathrm{e}^{\mathrm{i}sy})(x) = \lim_{\rho\to\infty}\exp\left[-nx\left(\frac{(n\rho-\mathrm{i}s)^{\rho}-(n\rho)^{\rho}}{2(n\rho-\mathrm{i}s)^{\rho}-(n\rho)^{\rho}}\right)\right]\cdot$$

$$\left(\frac{(n\rho - \mathrm{i}s)^{\rho}}{2(n\rho - \mathrm{i}s)^{\rho} - (n\rho)^{\rho}}\right)^{1+\alpha} = \frac{1}{(2 - \mathrm{e}^{\mathrm{i}s/n})^{1+\alpha}} \cdot$$

$$\exp\left(\frac{nx(\mathrm{e}^{\mathrm{i}s/n} - 1)}{\mathrm{e}^{\mathrm{i}s/n} - 2}\right) = (W_n^{\alpha}\,\mathrm{e}^{\mathrm{i}sy})(x)$$

where W_n^{α} is a positive linear operator based on Laguerre polynomials given by Eq.(4) with moment-generating function as indicated in Remark 2.

Following Theorem 1 of Ref.[20] and references therein, we get the desired outcomes.

Consider the well-known Szász-Mirakjan operator on the class $C[0,\infty)$ defined as

$$(S_n f)(x) = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-nx} \cdot (nx)^k}{k!} f\left(\frac{k}{n}\right)$$

Here, we introduce a new Păltănea operator $M_{n,\rho}^{\alpha}$ which arises as the combination of Szász-Mirakyan operator S_n with Laguerre Păltănea $L_{n,\rho}^{\alpha}$ on the space $C[0,\infty)$:

$$(M_{n,\rho}^{\alpha} f)(x) = (S_n \circ L_{n,\rho}^{\alpha} f)(x) =$$

$$\sum_{k=1}^{\infty} m_{n,k}^{\alpha}(x) \int_0^{\infty} s_{n,k}^{\rho}(y) f(y)\,\mathrm{d}y + m_{n,0}^{\alpha}(x) f(0)$$

where "$\circ$" represents the composition of two operation, and

$$m_{n,k}^{\alpha}(x) = \sum_{j=0}^{\infty} \frac{(nx)^j \cdot \mathrm{e}^{-(nx+\frac{j}{2})}}{j! \cdot 2^{\alpha+k+1}} \cdot L_k^{\alpha}\left(-\frac{j}{2}\right)$$

Lemma 2 For some real parameter A, the following equation holds:

$$(M_{n,\rho}^{\alpha} \exp_A)(x) =$$

$$\exp\left[nx\left(\exp\left(\frac{(n\rho)^{\rho} - (n\rho - A)^{\rho}}{2(n\rho - A)^{\rho} - (n\rho)^{\rho}}\right) - 1\right)\right] \cdot$$

$$\left[\frac{(n\rho - A)^{\rho}}{2(n\rho - A)^{\rho} - (n\rho)^{\rho}}\right]^{1+\alpha}$$

For certain non-zero constants a_i, $i = 0, 1, 2, \cdots$, we have

$$\left(M_{n,\rho}^{\alpha} \sum_{i\geqslant 0} a_i (e_1 - x e_0)^i\right)(x) = a_0 + a_1\left[\frac{1+\alpha}{n}\right] +$$

$$a_2\left[\frac{x}{n\rho}(4\rho + 1) + \frac{1}{n^2\rho}\left(\alpha^2\rho + 4\alpha\rho + 3\rho + \alpha + 1\right)\right] + \cdots$$

2 Some New Discrete Operators Based on Laguerre Polynomials

A new discrete operator can be evolved using the Laguerre-Păltănea and Szász-Mirakyan operators. The composition of Laguerre-Păltănea operators $L_{n,\rho}^{\alpha}$ with the Szász-Mirakyan operators S_n on some subspace of $C[0,\infty)$ for which $R_{n,\rho}^{\alpha}$ exists, can be defined as follows:

$$(R_{n,\rho}^{\alpha} f)(x) = ((L_{n,\rho}^{\alpha} \circ S_n) f)(x) = (L_{n,\rho}^{\alpha} S_n f)(x) =$$

$$\sum_{j=1}^{\infty} l_{n,j}^{\alpha}(x) \int_0^{\infty} \frac{n\rho}{\Gamma(j\rho)} \cdot \mathrm{e}^{-n\rho y} \cdot (S_n f)(y)\,\mathrm{d}y +$$

$$\frac{\mathrm{e}^{-\frac{nx}{2}}}{2^{\alpha+1}}(S_n f)(0) \tag{6}$$

Theorem 2 A concise form for $R_{n,\rho}^{\alpha}$ is provided by

$$(R_{n,\rho}^{\alpha} f)(x) = \sum_{k=0}^{\infty} b_{n,k}^{\rho,\alpha}(x) f\left(\frac{k}{n}\right)$$

where

$$b_{n,0}^{\rho,\alpha}(x) = \frac{\mathrm{e}^{-nx/2}}{2^{\alpha+1}}\left[\sum_{j=0}^{\infty}\left(\frac{1}{2}\left(\frac{\rho}{\rho+1}\right)^{\rho}\right)^{j}\binom{j+\alpha}{\alpha}\,{}_1F_1\left(-j; \alpha+1; \frac{-nx}{2}\right)\right]$$

and for $k \geqslant 1$,

$$b_{n,k}^{\rho,\alpha}(x) = \frac{\mathrm{e}^{-nx/2}}{2^{\alpha+1} \cdot (\rho+1)^k}\sum_{j=1}^{\infty}\frac{1}{2^j} \cdot \left(\frac{\rho}{1+\rho}\right)^{j\rho} \cdot \binom{j\rho+k-1}{k}\binom{\alpha+j}{\alpha}\,{}_1F_1\left(-j; 1+\alpha; \frac{-nx}{2}\right)$$

Proof Eq.(6) can be rewritten as:

$$(R_{n,\rho}^{\alpha} f)(x) = \frac{\mathrm{e}^{-nx/2}}{2^{\alpha+1}} \cdot \sum_{j=1}^{\infty}\left(\frac{1}{2^j} L_j^{\alpha}\left(-\frac{nx}{2}\right) \cdot \int_0^{\infty}\frac{(n\rho)^{j\rho} \cdot \mathrm{e}^{-n\rho y} \cdot y^{j\rho-1}}{\Gamma(j\rho)}\sum_{k=0}^{\infty}\frac{\mathrm{e}^{-ny}(ny)^k}{k!} f\left(\frac{k}{n}\right)\mathrm{d}y\right) + \frac{\mathrm{e}^{-nx/2}}{2^{\alpha+1}} \cdot f(0)$$

All the terms involved here, under sum and integration, are non-negative. Using Tonelli's theorem, we can express the above form as follows:

$$(R_{n,\rho}^{\alpha} f)(x) = \frac{\mathrm{e}^{-nx/2}}{2^{\alpha+1}}\sum_{k=0}^{\infty}\left(\frac{n^k}{k!} f\left(\frac{k}{n}\right) \cdot \sum_{j=1}^{\infty}\left(\frac{(n\rho)^{\rho}}{2}\right)^{j} \cdot \frac{1}{\Gamma(j\rho)} \cdot \binom{\alpha+j}{\alpha}\,{}_1F_1 \cdot \left(-j; 1+\alpha; -\frac{nx}{2}\right) \cdot I_{n,k}^{j,\rho}\right)$$

where

$$I_{n,k}^{j,\rho} = \int_0^{\infty} \mathrm{e}^{-ny(\rho+1)} \cdot y^{j\rho+k-1}\,\mathrm{d}y = \frac{\Gamma(j\rho+k)}{(n(\rho+1))^{j\rho+k}}$$

Thus

$$(R_{n,\rho}^{\alpha} f)(x) = \sum_{k=0}^{\infty}\frac{\mathrm{e}^{-\frac{nx}{2}}}{2^{\alpha+1} \cdot (1+\rho)^k} \cdot$$

$$f\left(\frac{k}{n}\right)\sum_{j=1}^{\infty}2^{-j}\cdot\left(\frac{\rho}{\rho+1}\right)^{j\rho}\cdot\binom{j\rho+k-1}{k}\cdot$$
$$\binom{\alpha+j}{\alpha}{}_1F_1\left(-j;\alpha+1;-\frac{nx}{2}\right)+\frac{\mathrm{e}^{-nx/2}}{2^{\alpha+1}}\cdot f(0)$$

Finally, the concise form for Eq.(6) is given by

$$(R_{n,\rho}^{\alpha}f)(x)=\sum_{k=0}^{\infty}b_{n,k}^{\rho,\alpha}(x)f\left(\frac{k}{n}\right)$$

where

$$b_{n,0}^{\rho,\alpha}(x)=\frac{e^{-nx/2}}{2^{\alpha+1}}\left[\sum_{j=0}^{\infty}\left(\frac{1}{2}\left(\frac{\rho}{\rho+1}\right)^{\rho}\right)^{j}\binom{j+\alpha}{\alpha}{}_1\cdot\right.$$
$$\left.F_1\left(-j;\alpha+1;-\frac{nx}{2}\right)\right]$$

and for $k\geqslant 1$,

$$b_{n,k}^{\rho,\alpha}(x)=\frac{\mathrm{e}^{-nx/2}}{2^{\alpha+1}\cdot(\rho+1)^{k}}\sum_{j=1}^{\infty}2^{-j}\cdot\left(\frac{\rho}{1+\rho}\right)^{j\rho}\cdot$$
$$\binom{j\rho+k-1}{k}\binom{\alpha+j}{\alpha}{}_1F_1\left(-j;1+\alpha;-\frac{nx}{2}\right)$$

Theorem 3 The following holds:

$$(R_{n,\rho}^{\alpha}\exp_A)(x)=$$
$$\exp\left[-nx\left(\frac{(\rho-\mathrm{e}^{A/n}+1)^{\rho}-\rho^{\rho}}{2(\rho-\mathrm{e}^{A/n}+1)^{\rho}-\rho^{\rho}}\right)\right]\cdot$$
$$\left[\frac{(\rho-\mathrm{e}^{A/n}+1)^{\rho}}{2(\rho-\mathrm{e}^{A/n}+1)^{\rho}-\rho^{\rho}}\right]^{1+\alpha}$$

Proof Taking into account Proposition 1, we obtain

$$(R_{n,\rho}^{\alpha}\exp_A)(x)=((L_{n,\rho}^{\alpha}\circ S_n)\exp_A)(x)=$$
$$(L_{n,\rho}^{\alpha}(S_n\exp_A))(x)=(L_{n,\rho}^{\alpha}\mathrm{e}^{ny(\mathrm{e}^{A/n}-1)})(x)=$$
$$\exp\left[-nx\left(\frac{(n\rho-n\mathrm{e}^{A/n}+n)^{\rho}-(n\rho)^{\rho}}{2(n\rho-n\mathrm{e}^{A/n}+n)^{\rho}-(n\rho)^{\rho}}\right)\right]\cdot$$
$$\left[\frac{(n\rho-n\mathrm{e}^{A/n}+n)^{\rho}}{2(n\rho-n\mathrm{e}^{A/n}+n)^{\rho}-(n\rho)^{\rho}}\right]^{1+\alpha}$$

Simplifying further, we get the desired results.

Corollary 1 For certain non-zero constants b_i, $i=0,1,2,\cdots$, we have

$$\left(R_{n,\rho}^{\alpha}\sum_{i\geqslant 0}b_i e_i\right)(x)=b_0+b_1\left[x+\frac{1+\alpha}{n}\right]+$$
$$b_2\left[x^2+\frac{2x}{n}\left(\alpha+\frac{1}{2\rho}+3\right)+\frac{1}{n^2\rho}(\alpha^2\rho+5\alpha\rho+\right.$$
$$\left.4\rho+\alpha+1)\right]+\cdots$$

The proof follows from the computations based on Theorem 3.

Theorem 4 For $f\in C_{\mathrm{BD}}(\mathbb{R}^+)$, $r\geqslant 1$, $s\in\mathbb{R}$ and $x\in[0,\infty)$, the following limits hold true:

1) $\lim\limits_{n\to\infty}(R_{rn,\rho}^{\alpha}f(ny))\left(\frac{x}{n}\right)=(R_{r,\rho}^{\alpha}f(y))(x)$

2) $\lim\limits_{n\to\infty}\left(R_{r,\rho}^{\alpha}f\left(\frac{y}{n}\right)\right)(nx)=f(x)$

Proof Using Theorem 3, we have

$$\lim_{n\to\infty}(R_{rn,\rho}^{\alpha}\mathrm{e}^{isny})\left(\frac{x}{n}\right)=$$
$$\lim_{n\to\infty}\exp\left[-rx\left(\frac{(\rho+1-\mathrm{e}^{is/r})^{\rho}-(\rho)^{\rho}}{2(\rho+1-\mathrm{e}^{is/r})^{\rho}-(\rho)^{\rho}}\right)\right]\cdot$$
$$\left[\frac{(\rho+1-\mathrm{e}^{is/r})^{\rho}}{2(\rho+1-\mathrm{e}^{is/r})^{\rho}-(\rho)^{\rho}}\right]^{1+\alpha}=(R_{r,\rho}^{\alpha}\mathrm{e}^{isy})(x)$$

Next

$$\lim_{n\to\infty}(R_{r,\rho}^{\alpha}\mathrm{e}^{\frac{isy}{n}})(nx)=$$
$$\lim_{n\to\infty}\exp\left[-rnx\left(\frac{(\rho-\mathrm{e}^{is/nr}+1)^{\rho}-\rho^{\rho}}{2(\rho-\mathrm{e}^{is/nr}+1)^{\rho}-\rho^{\rho}}\right)\right]\cdot$$
$$\left[\frac{(\rho-\mathrm{e}^{is/nr}+1)^{\rho}}{2(\rho-\mathrm{e}^{is/nr}+1)^{\rho}-\rho^{\rho}}\right]^{1+\alpha}=$$
$$\mathrm{e}^{isx}=\mathrm{Id}(\mathrm{e}^{isy},x)$$

Following Theorem 1 of Ref.[20] and references therein, we get the desired results.

Theorem 5[21] Let $f\in C[0,\infty)$, and $[a,b]$ is a closed and bounded subinterval of $[0,\infty)$. Then, for $x\in[a,b]$ and any positive linear operator L_n preserving constant functions, we have

$$|(L_nf)(x)-f(x)|\leqslant 2\,\omega_1(f,\delta_n(x))$$

where $\delta_n(x)=\sqrt{(L_n(t-x)^2)(x)}$.

Proof Consider

$$|(L_nf)(x)-f(x)|\leqslant(L_n|f(t)-f(x)|)(x)\leqslant$$
$$\left(L_n\left(1+\frac{(t-x)^2}{\delta^2}\right)\omega_1(f,\delta)\right)(x)$$

Choosing $\delta:=\delta_n(x)=\sqrt{(L_n(t-x)^2)(x)}$ on $[a,b]$, we get the required result.

Next, we produce two new discrete operators: $V_{n,\alpha}^{\mathrm{WS}}$, which is created by composing the Laguerre operator W_n^{α} proposed by Sucu et al.[4] and defined in Eq.(4) with the Szász-Mirakyan S_n; and $V_{n,\alpha}^{\mathrm{SW}}$, which is created by composing Szász-Mirakyan operator S_n with W_n^{α}, for the functions on the space $C[0,\infty)$ for which $V_{n,\alpha}^{\mathrm{WS}}$ and $V_{n,\alpha}^{\mathrm{SW}}$ exist respectively.

1) The first composition $V_{n,\alpha}^{\mathrm{WS}}$ has the following succinct form:

$$(V_{n,\alpha}^{\mathrm{WS}}f)(x)=\sum_{k=0}^{\infty}ws_{n,k}^{\alpha}(x)\cdot f\left(\frac{k}{n}\right)$$

where

$$ws_{n,k}^{\alpha}(x)=\frac{\mathrm{e}^{-\frac{nx}{2}}}{2^{\alpha+1}\cdot k!}\sum_{j=0}^{\infty}\frac{j^k}{(2\mathrm{e})^j}\cdot L_j^{\alpha}\left(-\frac{nx}{2}\right)$$

2) The next composition $V_{n,\alpha}^{\mathrm{SW}}$ has the succinct form given by:

$$(V_{n,\alpha}^{\mathrm{SW}} f)(x) = \sum_{k=0}^{\infty} sw_{n,k}^{\alpha}(x) \cdot f\left(\frac{k}{n}\right)$$

where

$$sw_{n,k}^{\alpha}(x) = \frac{\mathrm{e}^{-nx}}{2^{\alpha+k+1}} \sum_{j=0}^{\infty} \frac{(n^2 x^2 \mathrm{e}^{-1})^{\frac{j}{2}}}{j!} \cdot L_k^{\alpha}\left(-\frac{j}{2}\right)$$

Lemma 3 The moment-generating function for $V_{n,\alpha}^{\mathrm{WS}}$ and $V_{n,\alpha}^{\mathrm{SW}}$ are given by:

1) $$(V_{n,\alpha}^{\mathrm{WS}} \exp_A)(x) = \frac{1}{(2\mathrm{e} - \mathrm{e}^{\mathrm{e}^{\frac{A}{n}}})^{1+\alpha}} \cdot \exp\left[1 + \alpha - \frac{nx(\mathrm{e} - \mathrm{e}^{\mathrm{e}^{\frac{A}{n}}})}{2\mathrm{e} - \mathrm{e}^{\mathrm{e}^{\frac{A}{n}}}}\right]$$

2) $$(V_{n,\alpha}^{\mathrm{SW}} \exp_A)(x) = \frac{1}{(2 - \mathrm{e}^{\frac{A}{n}})^{1+\alpha}} \cdot \exp\left[nx\left(\exp\left(\frac{\mathrm{e}^{\frac{A}{n}} - 1}{2 - \mathrm{e}^{\frac{A}{n}}}\right) - 1\right)\right]$$

Proof Consider

$$(V_{n,\alpha}^{\mathrm{WS}} \exp_A)(x) = \sum_{k=0}^{\infty} l_{n,k}^{\alpha}(x) \cdot \mathrm{e}^{k(\mathrm{e}^{\frac{A}{n}}-1)} = (W_n^{\alpha} \mathrm{e}^{ny(\mathrm{e}^{\frac{A}{n}}-1)})(x)$$

Using Remark 2, we achieve the required result. Similary, we obtain the second outcome.

Lemma 4 For certain constants $c_i, i = 0,1,2,\cdots, c_i \neq 0$, we have

$$\left(V_{n,\alpha}^{\mathrm{WS}} \sum_{i \geqslant 0} c_i\, e_i\right)(x) = c_0 + c_1\left[x + \frac{1+\alpha}{n}\right] + c_2\left[x^2 + \frac{2x(\alpha+3)}{n} + \frac{\alpha^2 + 5\alpha + 4}{n^2}\right] + \cdots$$

and

$$\left(V_{n,\alpha}^{\mathrm{SW}} \sum_{i \geqslant 0} c_i\, e_i\right)(x) = c_0 + c_1\left[x + \frac{1+\alpha}{n}\right] + c_2\left[x^2 + \frac{2x(\alpha+3)}{n} + \frac{\alpha^2 + 4\alpha + 3}{n^2}\right] + \cdots$$

Proof follows from the calculation and Lemma 3. We skip the details.

Theorem 6 For $f \in K \subset C_{\mathrm{BD}}(\mathbb{R}^+)$, where K includes functions for which $V_{n,\alpha}^{\mathrm{WS}}$ and $V_{n,\alpha}^{\mathrm{SW}}$ both exist. If f'' exists at some point $x \in \mathbb{R}^+$, then we have

$$\lim_{n\to\infty} n[(V_{n,\alpha}^{\mathrm{WS}} f)(x) - f(x)] = \lim_{n\to\infty} n[(V_{n,\alpha}^{\mathrm{SW}} f)(x) - f(x)] = (1+\alpha) f'(x) + 2x f''(x)$$

Applying Lemma 4 with Taylor's expansion of f generates the proof. We omit the details.

Theorem 7 For $f \in C_{\mathrm{BD}}(\mathbb{R}^+)$ and $x \in \mathbb{R}^+$, we have

$$\lim_{\rho\to\infty}(R_{n,\rho}^{\alpha} f)(x) = (V_{n,\alpha}^{\mathrm{WS}} f)(x)$$

Proof Taking into consideration Lemma 3, we get

$$\lim_{\rho\to\infty}(R_{n,\rho}^{\alpha} \mathrm{e}^{isy})(x) = \lim_{\rho\to\infty}\exp\left[-nx\left(\frac{(\rho - \mathrm{e}^{is/n} + 1)^{\rho} - \rho^{\rho}}{2(\rho - \mathrm{e}^{is/n} + 1)^{\rho} - \rho^{\rho}}\right)\right] \cdot \left(\frac{(\rho - \mathrm{e}^{is/n} + 1)^{\rho}}{2(\rho - \mathrm{e}^{is/n} + 1)^{\rho} - \rho^{\rho}}\right)^{1+\alpha} = \left(\frac{1}{2\mathrm{e} - \mathrm{e}^{is/n}}\right)^{\alpha+1} \cdot \exp\left[1 + \alpha - \frac{nx(\mathrm{e} - \mathrm{e}^{\mathrm{e}^{is/n}})}{2\mathrm{e} - \mathrm{e}^{\mathrm{e}^{is/n}}}\right] = (V_{n,\alpha}^{\mathrm{WS}} \mathrm{e}^{isy})(x)$$

Following Theorem 1 of Ref. [20], we obtain the result.

Remark 3 The use of Lemma 3, Lemma 4 and Theorem 6 yields the following conclusion: the two operators, $V_{n,\alpha}^{\mathrm{WS}}$ and $V_{n,\alpha}^{\mathrm{SW}}$, are not equal on some subspace of $C[0,\infty)$ where both exist; rather, they are compositions of operators, namely, modified Laguerre W_n^{α} and Szász operator S_n, but in a different order. They have the same Asymptotic formula.

3 Numerical Comparison of Errors

We compare the convergence of the operators, namely Laguerre Păltănea $L_{n,\rho}^{\alpha}$, composition of Szász with Laguerre $M_{n,\rho}^{\alpha}$, and composition of Laguerre with Szász $R_{n,\rho}^{\alpha}$, and interpret the error quantitatively in view of Theorem 5. We denote $\delta_{n,\rho}^{\alpha}(x)$ to be the square root of the second central moment for the positive linear operators under consideration in this paper.

From Tables 1−3, it can be concluded that for the three operators $L_{n,\rho}^{\alpha}, M_{n,\rho}^{\alpha}$ and $R_{n,\rho}^{\alpha}$ defined on some subset of $C[0,\infty)$, the error term for $x \in [0,5] \subset [0,\infty)$ reduces when either n increases, ρ increases or α increases. However, by comparing the error values among the three above-mentioned operators, we may infer that in some closed and bounded subinterval of $[0,\infty)$, the convergence is better (or the error is smaller) for the Laguerre Păltănea operator $L_{n,\rho}^{\alpha}$, followed by the composition of Szász with Laguerre $M_{n,\rho}^{\alpha}$ and, finally, the composition of Laguerre with Szász $R_{n,\rho}^{\alpha}$. Similar comparisons can also be made for the other operators on contracted or extended subintervals of $[0,\infty)$ and for various other values of n, ρ and α.

Remark 4 Using Corollary 1, Lemma 2, and the observation made in Section 3, a conclusion can be drawn that we obtain two different operators $M_{n,\rho}^{\alpha}$ and $R_{n,\rho}^{\alpha}$ with different convergences properties, even though they are the composition of Laguerre-Păltănea with Szász in different orders.

Table 1 An upper bound $\delta_{n,\rho}^{\alpha}(x)$ on the error term for $L_{n,\rho}^{\alpha}$ on the subinterval $[0,5]$

n	ρ						α
	1	2	5	10	20	50	
1	4.89898	4.58258	4.38178	4.31277	4.27785	4.25676	0
5	2.09762	1.96469	1.88043	1.85149	1.83685	1.82800	1
10	1.47648	1.38384	1.32514	1.30499	1.29480	1.28864	2
100	0.45321	0.42438	0.40610	0.39983	0.39665	0.39473	5
500	0.20153	0.18866	0.18050	0.17770	0.17628	0.17542	10
1000	0.14319	0.13414	0.12840	0.12643	0.12544	0.12483	20

Table 2 An upper bound $\delta_{n,\rho}^{\alpha}(x)$ on the error term for $M_{n,\rho}^{\alpha}$ on the subinterval $[0,5]$

n	ρ						α
	1	2	5	10	20	50	
1	5.38516	5.09902	4.91935	4.85798	4.82701	4.80833	0
5	2.32379	2.20454	2.12979	2.10428	2.09141	2.08365	1
10	1.63707	1.55403	1.50200	1.48425	1.47530	1.46990	2
100	0.50537	0.47969	0.46360	0.45811	0.45534	0.45367	5
500	0.22498	0.21353	0.20635	0.20390	0.20267	0.20192	10
1000	0.15970	0.15164	0.14659	0.14486	0.14399	0.14347	20

Table 3 An upper bound $\delta_{n,\rho}^{\alpha}(x)$ on the error term for $R_{n,\rho}^{\alpha}$ on the subinterval $[0,5]$

n	ρ						α
	1	2	5	10	20	50	
1	5.47723	5.19615	5.01996	4.95984	4.92950	4.91121	0
5	2.34094	2.22261	2.14849	2.12321	2.11045	2.10276	1
10	1.64621	1.56365	1.51195	1.49432	1.48543	1.48007	2
100	0.50596	0.48031	0.46424	0.45876	0.45599	0.45433	5
500	0.22508	0.21363	0.20646	0.20401	0.20278	0.20203	10
1000	0.15977	0.15171	0.14666	0.14494	0.14407	0.14354	20

4 Graphical Comparison Between Operators

From Figs. 1-7, we analyze the convergence of the operators for a continuous function $f(x) = x \cdot e^{-x}$.

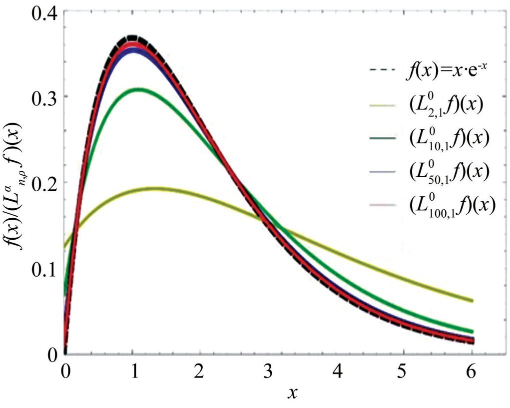

Fig.1 Convergence of $(L_{n,1}^{0}f)(x)$ for $f(x) = x \cdot e^{-x}$

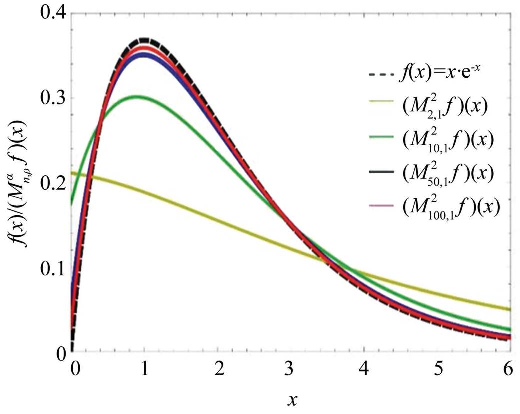

Fig.2 Convergence of $(M_{n,1}^{2}f)(x)$ for $f(x) = x \cdot e^{-x}$

Figs. 1-5 represent the convergence of operators $L_{n,\rho}^{\alpha}$, $M_{n,\rho}^{\alpha}$, $R_{n,\rho}^{\alpha}$, $V_{n,\alpha}^{\mathrm{WS}}$, and $V_{n,\alpha}^{\mathrm{SW}}$, respectively, for a continuous and infinitely differentiable function $f(x) =$

$x\cdot \mathrm{e}^{-x}$. We can conclude that for some specific values of $\rho > 0$ and $\alpha > -1$, increasing the value of n enhances the convergence of the related operator to the considered function. Figs. 6 and 7 compare the convergence of all the operators with $n = 10$ and 50. We may infer that the Laguerre-Păltănea operator $L^{\alpha}_{n,\rho}$ provides the best convergence. $L^{\alpha}_{n,\rho}$ is followed by $V^{\mathrm{SW}}_{n,\alpha}$, $V^{\mathrm{WS}}_{n,\alpha}$, $M^{\alpha}_{n,\rho}$ and finally $R^{\alpha}_{n,\rho}$. Similar comparisons can be made for various other functions and different values of parameters ρ and α .

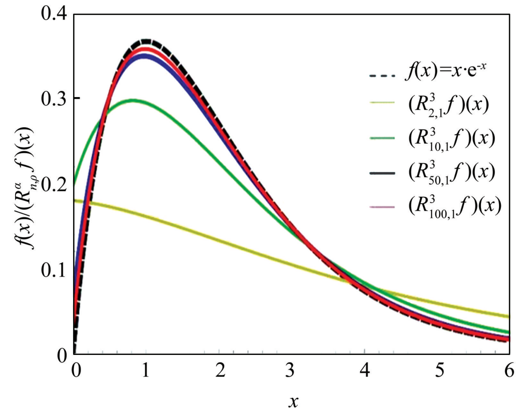

Fig.3 Convergence of $(R^3_{n,1}f)(x)$ for $f(x) = x\cdot \mathrm{e}^{-x}$

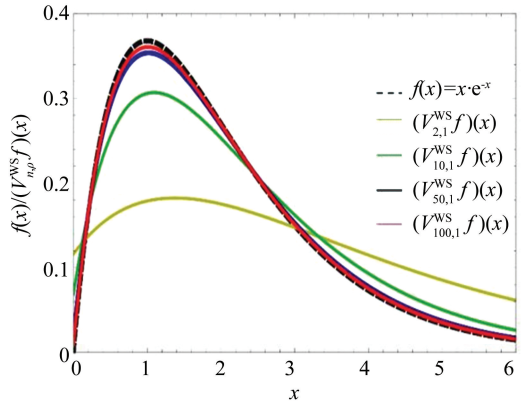

Fig.4 Convergence of $(V^{\mathrm{WS}}_{n,0}f)(x)$ for $f(x) = x\cdot \mathrm{e}^{-x}$

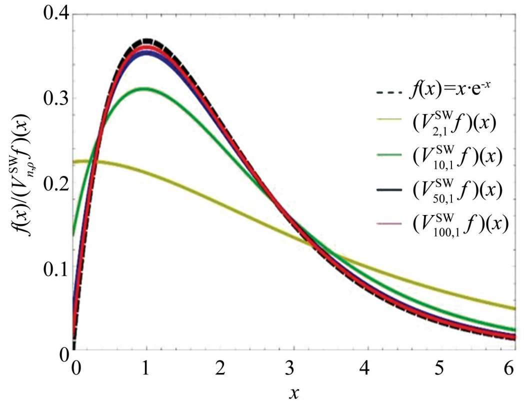

Fig.5 Convergence of $(V^{\mathrm{SW}}_{n,1}f)(x)$ for $f(x) = x\cdot \mathrm{e}^{-x}$

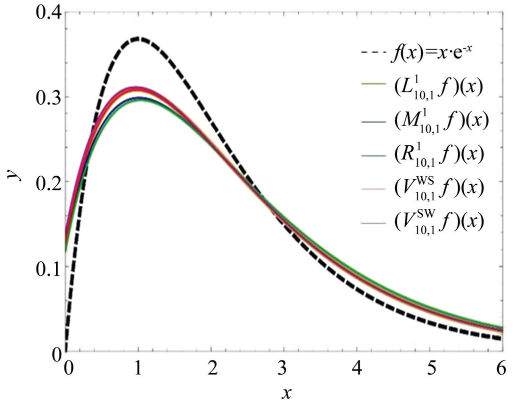

Fig.6 Convergence for $f(x) = x\cdot \mathrm{e}^{-x}$ for $n = 10$

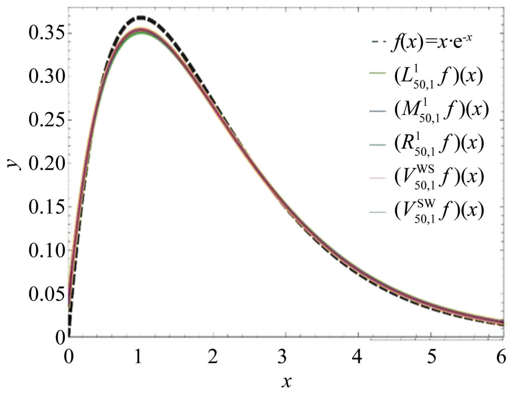

Fig.7 Convergence for $f(x) = x\cdot \mathrm{e}^{-x}$ for $n = 50$

5 Conclusions

In this study, we investigate approximation properties of Laguerre-type operators, with a particular focus on the Laguerre-Păltănea operator defined by modified Laguerre polynomials. We further constructed new families of integral and discrete operators by combining the Laguerre, Laguerre-Păltănea, and Szász-Mirakyan operators. Analytical results are complemented by numerical and graphical analyses using Mathematica software to examine convergence properties and approximation behavior.

From Tables 1-3, it can be concluded that for the operators $L^{\alpha}_{n,\rho}$, $M^{\alpha}_{n,\rho}$ and $R^{\alpha}_{n,\rho}$, the error decreases as the parameters n,ρ or α increase. Among these, the Laguerre-Păltănea operator $L^{\alpha}_{n,\rho}$ consistently provides better convergence, followed by the Szάsz-Laguerre composition $M^{\alpha}_{n,\rho}$ and, lastly, the Laguerre-Szάsz composition $R^{\alpha}_{n,\rho}$ in some compact subinterval of $[0,\infty)$.

The graphical analysis reinforces these observations. For the test function $f(x) = x\cdot \mathrm{e}^{-x}$, all operators provide good convergence as n increases, but the Laguerre-Păltnaea operator demonstrates the strongest approximation. The comparative graphs indicate the following hierarchy of performance: $L_{n,\rho}^{\alpha}$ exhibites the best convergence, followed by $V_{n,\alpha}^{\mathrm{SW}}$, $V_{n,\alpha}^{\mathrm{WS}}$, $M_{n,\rho}^{\alpha}$, and finally $R_{n,\rho}^{\alpha}$. Overall, our findings highlight that the Laguerre-Păltănea operator is the most effective among the considered operators in terms of convergence and error reduction. These results not only validate the effectiveness of this operator but also suggest new directions for further study of Laguerre-based approximation processes under different parameter regimes and for broader classes of functions.

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