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The Feasible Inexact Projected Extragradient Method for Solving Quasimonotone Variational Inequality Problems in Hilbert Space

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Abstract: In this study, a new extragradient algorithm is proposed, which combines an inexact projection operator with relative error to solve quasimonotone variational inequality problems in infinite-dimensional Hilbert space. The algorithm integrates a feasible inexact projection operator into the classical extragradient framework, and theoretical analysis shows that the method has weak convergence under the assumption that the operator is Lipschitz continuous. In addition, the strong convergence of the iterative sequence is guaranteed when the operator exhibits strong pseudomonotonicity. The effectiveness and practical performance of the algorithm are demonstrated through numerical experiments on typical problem instances. The proposed approach contributes to the advancement of variational inequality theory by extending the applicability of extragradient methods to broader classes of operators. It also provides a scalable and efficient solution paradigm for large-scale optimization problems involving quasimonotone structures.

Keywords: variational inequality problem, inexact projection, quasimonotone mapping, strong convergence

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0 Introduction

The variational inequality problem is formulated as follows: In the context of a real Hilbert space H, given a nonempty subset $C \subset H$ that is closed and convex, consider a mapping $F: H \to H$, we seek an element $x \in C$ such that

$$\langle F(x), y - x\rangle \geqslant 0, \forall y \in C \tag{1}$$

We abbreviate this problem as VIP[1]. The set of all solutions to problem (1) is denoted by S, and is assumed to be nonempty. The set S_D, which consists of all feasible solutions, is defined by the subsequent mathematical problem,

$$\langle F(y), y - x\rangle \geqslant 0, \forall y \in C \tag{2}$$

Obviously, the set S_D is both closed and convex, and $S_D \subset S$, For this study, we assume that it is also nonempty. Given the continuity of the operator F and the convexity of C, if F also satisfies the condition of being pseudomonotone, then the solution sets S and S_D coincide, as demonstrated in Lemma 2.1 from Ref.[2]. But the conclusion $S \subset S_D$ fails to remain true when the operator is quasimonotone and exhibits continuity.

To address variational inequality problems, a range of projection-based methods has been introduced and rigorously examined in previous studies [3-12]. We operate in the context of a real Hilbert space H, and there exists a constraint set C that satisfies three fundamental properties: it is not empty, closed and convex. The mapping that identifies the unique point in C closest to a given vector $x \in H$ is formally defined as the projection operator P_C. It is defined by the following procedure:

$$P_C(x) = \operatorname{argmin}\{\|y - x\| : y \in C\} \tag{3}$$

Problem (1) may also be interpreted through a fixed-point perspective, namely, its resolution corresponds to locating a point of the type: find $x^* \in C$ such that $x^* = P_C(x^* - \lambda F(x^*))$, for any $\lambda > 0$.

By applying the previously introduced formula, we are able to derive the following iterative scheme, commonly referred to as the projected gradient

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method. Addressing variational inequality problems often leads researchers to the Projected Gradient Method (PGM), which typically stands out due to its simplicity and widespread adoption[13], as detailed below:

$$\xi_{t+1} = P_C(\xi_t - \lambda F(\xi_t))$$

Assuming F satisfies strongly monotonicity with modulus η together with L-Lipschitz continuity, then taking any positive scalar step λ ensures that the iterates generated by the scheme approach exhibit strong convergence properties, ultimately arriving at the sole solution to the given variational inequality. These strict conditions greatly limit the scope of application of this method in practical problems. If F only holds monotonicity and Lipschitz continuity, the PGM is not ensured to converge, as noted in Ref. [14]. Korpelevich and Antipin[15-16] proposed an algorithm to tackle VIP in finite-dimensional space, the scheme known as Extragradient Method (EGM) can be reformulated as follows:

$$\varrho_t = P_C(\xi_t - \lambda F(\xi_t))$$
$$\xi_{t+1} = P_C(\xi_t - \lambda F(\varrho_t))$$

It relies on an operator F that exhibits monotonicity together with L-Lipschitz continuity, and its iterative updates are performed with a positive parameter controlling the step length λ, which is chosen from the interval $(0, 1/L)$.

In contexts of variational inequality problems where the map only satisfies quasimonotone, the convergence of the interior neighborhood method, as discussed in Refs. [11-12], can be established under alternative conditions, such as the nonemptiness of the set S_{D}. In the study of Ye and He[3], a novel dual-projection scheme was developed to address variational inequalities with quasimonotone (or potentially non-monotone) operators in finite-dimensional Euclidean spaces $\mathbb{R}^n$ with the key precondition that the solution set S_{D} is nonempty.

Typically, each iteration requires solving two constrained projection subproblems over the feasible domain. This computational burden becomes particularly pronounced when dealing with high-dimensional systems, where the increased number of variables may substantially elevate per-iteration expenses. Therefore, when the method iteratively deviates from the solution of the considered problem, it may not be reasonable to perform an exact projection. In recent decades, researchers[12, 17-23] have successively devised various inexact methods to reduce the computational cost of projection.

Millán et al.[24] introduced a new iterative scheme employing practically feasible inexact projection techniques that incorporates the extragradient method to tackle variational inequalities (1) with pseudomonotone structure in Euclidean space of limited dimensionality, building on more sophisticated projection techniques proposed by many researchers[3-10, 25]. Similar to the structure of traditional extragradient algorithms, this method replaces exact projections with inexact projections by the following procedure:

$$\varrho_t \in P_C^{\gamma_t}(\xi_t, \xi_t - rF(\xi_t))$$
$$\xi_{t+1} \in P_C^{\gamma_t}(\xi_t, \xi_t - rF(\varrho_t))$$

where the mapping F is pseudomonotone, L-Lipschitz continuous. The step size r satisfies $0 \leqslant r \leqslant \sqrt{1 - 2\bar{\gamma}}/L$, and the error tolerance γ_n satisfies $0 \leqslant \gamma_t \leqslant \bar{\gamma}$. The inexact projection P_C^{γ} is defined as

$$P_C^{\gamma}(a, \tau) := \Big\{ l \in C : \langle \tau - l, \varpi - l \rangle \leqslant \gamma \| l - a \|^2, \forall\, \varpi \in C \Big\} \tag{4}$$

Motivated by this line of research, the present study generalizes the extragradient inexact projection method with a tolerable error to address a category of VIPs characterized by quasimonotonicity in Hilbert space. Theoretical analysis demonstrates that, within a Hilbert space of infinite dimension, if the operator exhibits quasimonotone behavior and satisfies the Lipschitz continuity condition, the method achieves weak convergence, aligning with the convergence behavior of existing algorithms[16, 26]. Moreover, assuming that the mapping exhibits δ-strong pseudomonotone behavior within a Hilbert space, the iterative method constructs a series of points whose limit is a solution to the variational inequality, with the convergence being of a strong nature.

The structure of this study is as follows: Section 1 outlines fundamental concepts and assembles the foundational theorems essential to proving the convergence results. Section 2 details an extragradient algorithm based on inexact projection, along with a rigorous proof of its weak and strong convergence under predefined conditions. Section 3 demonstrates the algorithm's practical performance and effectiveness through numerical experiments on typical problem instances. Conclusions are presented in Section 4.

1 Preliminaries

Let H be a real Hilbert space characterized by inner products and norms, indicated as $\langle \cdot, \cdot \rangle$, and $\| \cdot \|$, respectively. To simplify notation, the concept of strong convergence is expressed using the symbol $\rightarrow$, whereas the presence of weak convergence is reflected through $\rightharpoonup$.

Assume we have a subset C residing in a real Hilbert space H. This set C is nonempty and exhibits both convexity and the property of being closed. The lemma below highlights some fundamental characteristics of metric projection P_C, frequently utilized in related mathematical analyses.

Lemma 1[27] For all $\tau, v \in H$, and $\eta \in C$, the following statements hold:

(a) $\langle \tau - P_C\tau, P_C\tau - \eta \rangle \geqslant 0$. $\|P_C\tau - P_Cv\| \leqslant \|\tau - v\|$.

(b) P_C is firmly nonexpansive, i. e., $\|P_C\tau - P_Cv\|^2 \leqslant \|\tau - v\|^2 - \|(\tau - P_C\tau) - (v - P_Cv)\|^2$, in particular, $\|P_C\tau - \eta\|^2 \leqslant \|\tau - \eta\|^2 - \|\tau - P_C\tau\|^2$.

(c) A is δ-strongly pseudomonotone on H, if there exists $\delta > 0$ such that $\langle Av, \tau - v \rangle \geqslant 0 \Rightarrow \langle A\tau, \tau - v \rangle \geqslant \delta \| \tau - v \|^2$.

(d) A is pseudomonotone on H, if $\langle A\tau, v - \tau \rangle \geqslant 0 \Rightarrow \langle Av, v - \tau \rangle \geqslant 0$.

(e) A is quasimonotone on H, if $\langle A\tau, v - \tau \rangle > 0 \Rightarrow \langle Av, v - \tau \rangle \geqslant 0$.

(f) A is L-Lipschitz continuous on H, if a constant $L > 0$ exists such that the inequality $\| A\tau - Av \| \leqslant L \| \tau - v \|$ holds.

(g) A is sequentially weakly continuous, i.e., if $\{\tau_k\} \rightharpoonup \tau$, then $\{A(\tau_k)\} \rightharpoonup A(\tau)$.

Obviously, $(c) \Rightarrow (d) \Rightarrow (e)$. Nevertheless, the reverse implications are not guaranteed in general.

This subsection revisits the notion of feasible inexact projections over closed convex domains, building upon foundational work documented in Refs. [19-20, 24-25, 28]. This study incorporates several novel characteristics of feasible inexact projections in Ref. [24], which serve as fundamental components for our subsequent analysis. The formal characterization of these approximate projections is presented below.

Definition 1 (Feasible inexact projection[24]) Consider a convex subset C that is also closed within a Hilbert space H and a nonnegative real parameter $\gamma (\gamma \geqslant 0)$ representing a given error tolerance of projection accuracy. Let $P_C^{\gamma}(a, \xi)$ denote a γ-feasible inexact projection operator of ξ taken relative to reference point $a \in C$ onto C, which is characterized by Eq.(4).

Feasible inexact projection $P_C^{\gamma}(a, \xi): H \rightrightarrows C$ is the set-valued mapping. This framework extends conventional projection theory. Below, we analyze key properties of this generalized operator.

Lemma 2[24] Given parameters $\gamma \geqslant 0$ be error tolerance, set $C \subset H$, $\xi \in H$, $a \in C$ and $\gamma > 0$, then

(a) When $\gamma = 0$, $P_C^0(a, \xi)$ coincides with the exact projection of ξ onto C, for all $\xi \in H$.

(b) The inclusion $P_C^0(a, \xi) \subset P_C^{\gamma}(a, \xi)$ guarantees $P_C^{\gamma}(a, \xi) \neq \emptyset$.

(c) Monotonicity: If $a \leqslant b$ then $P_C^a(a, \xi) \subset P_C^b(a, \xi)$.

(d) An analogous property to the firm non-expansiveness characteristic of the exact projection operator is established for the feasible inexact projection, and its proof is derived by employing a similar line of reasoning as that found in Ref. [24]. If $l_1 \in P_C^{\gamma}(a, \bar{\xi})$ and $l_2 = P_C(\bar{\xi})$, then

$$\| l_1 - l_2 \|^2 \leqslant \| \xi - \bar{\xi} \|^2 - \| (\xi - \bar{\xi}) - (l_1 - l_2) \|^2 + 2\gamma \| l_1 - a \|^2$$

Definition 2 Suppose S is a nonempty subset of the Hilbert space H. A sequence $\{\xi_t\} \subset H$ is said to be quasi-Fejér convergent to S, if and only if, for all $\xi \subset C$ there exists $t_1 \geqslant 0$ and a sequence $\{\epsilon_t\}$ with $\sum_{t=0}^{\infty} \epsilon_t < \infty$, such that $\|\xi_{t+1} - \xi\|^2 \leqslant \|\xi_t - \xi\|^2 + \epsilon_t$ for all $t \geqslant t_1$.

Lemma 3[29] Consider the sequence $\{\xi_t\} \subset H$ and $C \subset H$ be a nonempty set, and assume that the sequence $\{\xi_t\}$ demonstrates quasi-Fejér convergence towards C. The following two key outcomes are derived from the analysis:

1) The global behavior of the iterative process, confirming that the sequence $\{\xi_t\}$ is bounded.

2) The entire sequence $\{\xi_t\}$ converges to $\bar{\xi}$, if a cluster point $\bar{\xi}$ of $\{\xi_t\}$ lies in C.

Lemma 4[24] Let $F: H \rightarrow H$ be an operator defined over a Hilbert space. Suppose $C \subset H$ be a closed, convex set that is not empty. Consider $\xi_0 \in C$, $\gamma \in [0, 1)$ and $\alpha > 0$. If the points $\eta \in P_C^{\gamma}(\xi_0, \xi_0 - \lambda F(\xi_0))$ and $\xi_1 \in P_C^{\gamma}(\xi_0, \xi_0 - \lambda F(\eta))$, then the following estimates hold:

1) $\| \eta - \xi_0 \| \leqslant \dfrac{\lambda}{1 - \gamma} \| F(\xi_0) \|$

2) $\| \xi_1 - \xi_0 \| \leqslant \frac{\lambda}{1-\gamma} \| F(\eta) \|$

Moreover, if the mapping F satisfies Lipschitz continuity on C with Lipschitz constant $L > 0$, then the bound improves to:

$$\| \xi_1 - \xi_0 \| \leqslant \frac{\lambda(1-\gamma+\lambda L)}{(1-\gamma)^2} \| F(\xi_0) \|$$

Lemma 5 [24] The following characterizations of those points which satisfy the variational inequality problem stipulated by the operator F within the confines of the admissible set C are interchangeable:

1) The point ξ is contained in, which denotes the collection S of all solutions to the associated variational inequality VIP(F,C) .

2) $\xi \in P_C^{\gamma}(\xi,\xi - \lambda F(\xi))$, for all $\lambda \geqslant 0$.

3) There exists $\bar{h} > 0$, such that $\langle F(\xi), \zeta(\bar{h}) - \xi \rangle \geqslant 0$, for $\zeta(\bar{h}) \in P_C^{\gamma}(\xi,\xi - \bar{h}F(\xi))$.

Lemma 6 Let $\{\xi_t\} \subset H$ be a sequence that converges weakly to some element ξ ,then for any $y \in H$, with $y \neq \xi$, the minimal asymptotic distances satisfy:

$$\liminf_{t\to\infty} \| \xi_t - \xi \| < \liminf_{t\to\infty} \| \xi_t - y \| \qquad (5)$$

Lemma 7 [30] Consider a sequence $\{\phi_t\}$ comprised of nonnegative real values that fulfill the condition, where $\varphi_{t+1} \leqslant \phi_t \varphi_t + \psi_t$, $\forall\, t \in \mathbb{N}$, where ϕ_t and ψ_t are sequences formed by nonnegative numbers such that $\{\phi_t\} \subseteq [1, +\infty)$, $\{\psi_t\} \subseteq [0, +\infty)$, $\sum_{t=1}^{\infty} (\phi_t - 1) < \infty$ and $\sum_{t=1}^{\infty} \psi_t < \infty$. Then, we obtain that $\lim_{t\to\infty} \varphi_t$ exists.

2 Convergence Analysis

The current section introduces a new iterative framework designed for solving variational inequality problems (1) with quasimonotonicity assumptions in Hilbert spaces. Distinct from traditional projection-based methods, this scheme integrates feasible approximations of projections, guided by a suitably chosen tolerance condition involving relative error. Instead of requiring exact projections onto an acceptable range at each step, the algorithm permits controlled inaccuracies, improving computational efficiency while maintaining convergence.

We make some assumptions as follows:

(A1) The set S_{D} is nonempty: $S_{\mathrm{D}} \neq \emptyset$.

(A2) The mapping F is quasimonotone and L-Lipschitz continuous.

(A3) The mapping F is weakly sequentially lower semicontinuous, that is, whenever $\{\xi_t\} \subset H$ and $\xi_t \rightharpoonup \xi$, it holds that $F(\hat{\xi}) \leqslant \liminf_{t\to\infty} \| F(\xi_t) \|$.

(A4) The set $A = \{z \in C: F(z) = 0\} \setminus S_{\mathrm{D}}$ is a finite set.

(A5) The step size λ satisfies $0 < \lambda < \sqrt{1 - 2\bar{\gamma}}/L$.

(A6) $\sum_{t\in\mathbb{N}} \varepsilon_t < +\infty$.

Building upon the foundational extragradient framework, this study introduces an efficient variant that relaxes the requirement for exact projections. The formal procedural steps of this approach, designated as Algorithm 1, are outlined below:

Algorithm 1

Initialization: Choose parameters $\lambda > 0$, $\bar{\gamma} \in (0, 1/2)$, and $\sum_{t\in\mathbb{N}} \varepsilon_t < +\infty$.

Step 1: Select the initial point $\xi_1 \in C$, and set iteration counter $t := 1$.

Step 2: Determine the error tolerance γ_t satisfying:

$$0 \leqslant \gamma_t \| F(\xi_t) \|^2 \leqslant \varepsilon_t, 0 \leqslant \gamma_t < \bar{\gamma} \qquad (6)$$

Step 3: Calculate feasible inexact projections onto C as follows:

$$\varrho_t \in P_C^{\gamma_t}(\xi_t, \xi_t - \lambda F(\xi_t)) \qquad (7a)$$

$$\xi_{t+1} \in P_C^{\gamma_t}(\xi_t, \xi_t - \lambda F(\varrho_t)) \qquad (7b)$$

Stopping rule: If either $\xi_t = \varrho_t$ or $\varrho_t = \xi_{t+1}$, terminate the algorithm. Otherwise, increment $t := t + 1$ and return to Step 2.

It is worth noting that from inequality (6), we have

$$\sum_{t\in\mathbb{N}} \gamma_t \| F(\xi_t) \|^2 \leqslant +\infty, 0 \leqslant \gamma_t < \bar{\gamma} \qquad (8)$$

Consider an iterative process where the first iterate ξ_1 is selected from the set C and for every $t \in \mathbb{N}$, and each subsequent point ξ_{t+1} is constructed via a valid inexact projection into C that remains within the feasible domain. Under these conditions, the entire sequence $\{\xi_t\}$ is confined to C . Furthermore, given that C possesses the topological property of being closed, from the aforementioned properties, we can deduce the following: assuming the sequence $\{\xi_t\}$ possesses cluster points, each of which is inevitably contained within the specified solution set C , ensuring feasibility throughout.

To lay the groundwork for the primary theorem, we first deduce a key result that is fundamental to its development.

Lemma 8 Let $\{\xi_t\}$ and $\{\varrho_t\}$ be the sequence produced through the execution of Algorithm 1. The ensuing results are derived from these sequences: for $\widehat{\xi} \in S_{\mathrm{D}}$, $\lim_{k\to\infty}\|\xi_t - \widehat{\xi}\|$ exists, $\lim_{t\to\infty}\|\xi_t - \varrho_t\| = 0$, and $\lim_{t\to\infty}\|\xi_{t+1} - \xi_t\| = 0$.

Proof We set $z_t = \xi_t - \lambda F(\varrho_t)$, for $\widehat{\xi} \in S_{\mathrm{D}}$, we have

$$\|\xi_{t+1} - \widehat{\xi}\|^2 = \|\xi_{t+1} - z_t\|^2 + \|z_t - \widehat{\xi}\|^2 - 2\langle \xi_{t+1} - z_t, \widehat{\xi} - z_t\rangle = - \|\xi_{t+1} - z_t\|^2 + \|z_t - \widehat{\xi}\|^2 + 2\langle z_t - \xi_{t+1}, \widehat{\xi} - \xi_{t+1}\rangle$$

Since $\xi_{t+1} \in P_C^{\gamma_t}(\xi_t, z_t)$, for $\widehat{\xi} \in S_{\mathrm{D}} \subset C$, $\langle z_t - \xi_{t+1}, \widehat{\xi} - \xi_{t+1}\rangle \leqslant \gamma_t \|\xi_{t+1} - \xi_t\|^2$, we get

$$\|\xi_{t+1} - \widehat{\xi}\|^2 \leqslant \|z_t - \widehat{\xi}\|^2 - \|\xi_{t+1} - z_t\|^2 + 2\gamma_t \|\xi_{t+1} - \xi_t\|^2$$

According to the definition of $\{z_t\}$,

$$\|z_t - \widehat{\xi}\|^2 - \|\xi_{t+1} - z_t\|^2 = \|\xi_t - \lambda F(\varrho_t) - \widehat{\xi}\|^2 - \|\xi_t - \xi_{t+1} - \lambda F(\varrho_t)\|^2 = \|\xi_t - \widehat{\xi}\|^2 - \|\xi_t - \xi_{t+1}\|^2 + 2\lambda\langle F(\varrho_t), \widehat{\xi} - \xi_{t+1}\rangle$$

So

$$\|\xi_{t+1} - \widehat{\xi}\|^2 \leqslant \|\xi_t - \widehat{\xi}\|^2 - \|\xi_t - \xi_{t+1}\|^2 + 2\lambda\langle F(\varrho_t), \widehat{\xi} - \xi_{t+1}\rangle + 2\gamma_t \|\xi_{t+1} - \xi_t\|^2 \tag{9}$$

Since $\widehat{\xi} \in S_{\mathrm{D}}$, for $\varrho_t \in C$, we obtain $\langle F(\varrho_t), \varrho_t - \widehat{\xi}\rangle \geqslant 0$, then

$$\langle F(\varrho_t), \widehat{\xi} - \xi_{t+1}\rangle \leqslant \langle F(\varrho_t), \varrho_t - \xi_{t+1}\rangle$$

Noticing that $\|\xi_t - \xi_{t+1}\|^2 = \|\xi_t - \varrho_t\|^2 + \|\varrho_t - \xi_{t+1}\|^2 + 2\langle \xi_t - \varrho_t, \varrho_t - \xi_{t+1}\rangle$. We have

$$\|\xi_{t+1} - \widehat{\xi}\|^2 \leqslant \|\xi_t - \widehat{\xi}\|^2 - \|\xi_t - \varrho_t\|^2 - \|\varrho_t - \xi_{t+1}\|^2 + 2\gamma_t \|\xi_{t+1} - \xi_t\|^2 + 2\langle \xi_t - \lambda F(\varrho_t) - \varrho_t, \xi_{t+1} - \varrho_t\rangle \tag{10}$$

Moreover,

$$\langle \xi_t - \lambda F(\varrho_t) - \varrho_t, \xi_{t+1} - \varrho_t\rangle = \langle \xi_t - \lambda F(\xi_t) - \varrho_t, \xi_{t+1} - \varrho_t\rangle + \lambda\langle F(\xi_t) - F(\varrho_t), \xi_{t+1} - \varrho_t\rangle \tag{11}$$

According to the definition of $\{\xi_t\}$ and $\{\varrho_t\}$, then using Eq. (4), we get

$$\langle \xi_t - \lambda F(\xi_t) - \varrho_t, \xi_{t+1} - \varrho_t\rangle \leqslant \gamma_t \|\xi_t - \varrho_t\|^2 \tag{12}$$

Substituting Eq. (11) and inequality (12) into inequality (10), we have

$$\|\xi_{t+1} - \widehat{\xi}\|^2 \leqslant \|\xi_t - \widehat{\xi}\|^2 - \|\xi_t - \varrho_t\|^2 - \|\varrho_t - \xi_{t+1}\|^2 + 2\gamma_t \|\xi_{t+1} - \xi_t\|^2 + 2\gamma_t \|\varrho_t - \xi_t\|^2 + 2\lambda\langle F(\xi_t) - F(\varrho_t), \xi_{t+1} - \varrho_t\rangle \tag{13}$$

In addition, $\langle F(\xi_t) - F(\varrho_t), \xi_{t+1} - \varrho_t\rangle \leqslant L\|\xi_t - \varrho_t\|\|\xi_{t+1} - \varrho_t\|$, combined with the inequality (13), yields

$$\|\xi_{t+1} - \widehat{\xi}\|^2 \leqslant \|\xi_t - \widehat{\xi}\|^2 - \|\xi_t - \varrho_t\|^2 - \|\varrho_t - \xi_{t+1}\|^2 + 2\gamma_t \|\xi_{t+1} - \xi_t\|^2 + 2\gamma_t \|\varrho_t - \xi_t\|^2 + 2\lambda L\|\xi_t - \varrho_t\|\|\xi_{t+1} - \varrho_t\| \tag{14}$$

From Lemma 4, we can get

$$\|\xi_{t+1} - \widehat{\xi}\|^2 \leqslant \|\xi_t - \widehat{\xi}\|^2 - (1 - \lambda L - 2\gamma_t)\|\xi_t - \varrho_t\|^2 - (1 - \lambda L)\|\xi_{t+1} - \varrho_t\|^2 + \frac{2\lambda^2(1 - \bar{\gamma} + \lambda L)^2}{(1 - \bar{\gamma})^4}\gamma_t \|F(\xi_t)\|^2 \tag{15}$$

Set

$$\eta_1 = 1 - \lambda L - 2\bar{\gamma}, \eta_2 = 1 - \lambda L$$

$$\vartheta_1 = \frac{2\lambda^2(1 - \bar{\gamma} + \lambda L)^2}{(1 - \bar{\gamma})^4}$$

We have

$$\|\xi_{t+1} - \widehat{\xi}\|^2 \leqslant \|\xi_t - \widehat{\xi}\|^2 - \eta_1 \|\xi_t - \varrho_t\|^2 - \eta_2 \|\xi_{t+1} - \varrho_t\|^2 + \vartheta_1 \gamma_t \|F(\xi_t)\|^2 \tag{16}$$

Then, considering that $0 \leqslant \gamma_t < \bar{\gamma}$, and assumptions $0 < \lambda < \sqrt{1 - 2\bar{\gamma}}/L$, therefore $\eta_1 > 0$, $\eta_2 > 0$. Thus, it follows from inequality (16) that

$$\|\xi_{t+1} - \widehat{\xi}\|^2 \leqslant \|\xi_t - \widehat{\xi}\|^2 + \vartheta_1 \gamma_t \|F(\xi_t)\|^2$$

According to Definition 2, the inequality stated above indicates that the sequence $\{\xi_t\}$ exhibits quasi-Fejér convergent toward the S_{D}. Assuming that S_{D} is nonempty, this conclusion directly follows from term 1) of Lemma 3, we imply that $\{\xi_t\}$ is bounded. Then by Lemma 7, inequality (6) and assumptions $\sum_{t\in\mathbb{N}} \varepsilon_t < +\infty$, we obtain that $\lim_{t\to\infty}\|\xi_t - \widehat{\xi}\|$ exists.

Then from inequality (16), sum both sides simultaneously, we obtain

$$\eta_2 \|\xi_{t+1} - \varrho_t\|^2 \leqslant \|\xi_t - \widehat{\xi}\|^2 - \|\xi_{t+1} - \widehat{\xi}\|^2 + \vartheta_1 \gamma_t \|F(\xi_t)\|^2$$

We can get

$$\eta_2 \sum_{t=1}^{\infty} \|\xi_{t+1} - \varrho_t\|^2 \leqslant \|\xi_1 - \widehat{\xi}\|^2 + \vartheta_1 \sum_{t=1}^{\infty} \gamma_t \|F(\xi_t)\|^2 < +\infty$$

Then

$$\lim_{t\to\infty} \|\xi_{t+1} - \varrho_t\|^2 = 0$$

Using a similar approach, we have $\lim_{t\to\infty}\|\xi_t - \varrho_t\|^2 = 0$. According to $\|\xi_{t+1} - \xi_t\| \leqslant \|\xi_{t+1} - \varrho_t\| + \|\varrho_t - \xi_t\|$, therefore

$$\lim_{t\to\infty} \|\xi_{t+1} - \xi_t\|^2 = 0$$

Lemma 9 Let $\{\xi_t\}$ be a sequence produced through the execution of Algorithm 1. The ensuing results are derived from these sequences: if $\widehat{\xi}$ is a weak clustering point of $\{\xi_t\}$ and $\lim\limits_{t\to\infty}\|\xi_t - \varrho_t\|^2 = 0$, then $\widehat{\xi} \in S_{\mathrm{D}}$ or $F(\widehat{\xi}) = 0$.

Proof: According to Lemma 8, consider $\{\xi_t\}$ to be bounded, then it is possible to extract a subsequence $\{\xi_{t_n}\}$ from $\{\xi_t\}$ for which the limit condition holds $\xi_{t_n} \rightharpoonup \widehat{\xi} \in C$. There are two cases below.

Case I Assume the following is true.

$$\limsup_{n\to\infty}\|F(\xi_{t_n})\| = 0$$

$$\lim_{n\to\infty}\|F(\xi_{t_n})\| = \liminf_{n\to\infty}\|F(\xi_{t_n})\| = 0$$

Then

$$0 \leqslant \|F(\widehat{\xi})\| \leqslant \liminf_{n\to\infty}\|F(\xi_{t_n})\| = 0, \|F(\widehat{\xi})\| = 0$$

Case II Assume the following is true, $\limsup\limits_{n\to\infty}\|F(\xi_{t_n})\| > 0, \lim\limits_{n\to\infty}\|F(\xi_{t_n})\| = M_1 \geqslant 0$. We may assume, without sacrificing generality, that the subsequence of $\{F(\xi_{t_n})\}$ is itself, as the reasoning for a proper subsequence follows analogously. According to $\lim\limits_{t\to\infty}\|\xi_t - \varrho_t\| = 0$, then $\lim\limits_{n\to\infty}\|F(\varrho_{t_n})\| = M_1 \geqslant 0$. This implies the existence of a threshold index, denoted $t_0 \in \mathbb{N}$, starting from which the following condition holds universally: for each subsequent term, $\|F(\varrho_{t_n})\| > M_1/2$ holds.

Then according to $\varrho_{t_n} \in P_C^{\gamma_{t_n}}(\xi_{t_n}, \xi_{t_n} - \lambda F(\xi_{t_n}))$, so

$$\langle \xi_{t_n} - \lambda F(\xi_{t_n}) - \varrho_{t_n}, z - \varrho_{t_n}\rangle \leqslant \gamma_{t_n}\|\xi_{t_n} - \varrho_{t_n}\|^2, \forall z \in C$$

Divide both sides of the inequality by λ,

$$\frac{1}{\lambda}\langle \xi_{t_n} - \varrho_{t_n}, z - \varrho_{t_n}\rangle \leqslant \frac{\gamma_{t_n}}{\lambda}\|\xi_{t_n} - \varrho_{t_n}\|^2 + \langle F(\xi_{t_n}), z - \varrho_{t_n}\rangle, \forall z \in C$$

Rearranging the inequality gives

$$\frac{1}{\lambda}\langle \xi_{t_n} - \varrho_{t_n}, z - \varrho_{t_n}\rangle - \langle F(\xi_{t_n}) - F(\varrho_{t_n}), z - \varrho_{t_n}\rangle \leqslant \frac{\gamma_{t_n}}{\lambda}\|\xi_{t_n} - \varrho_{t_n}\|^2 + \langle F(\varrho_{t_n}), z - \varrho_{t_n}\rangle, \forall z \in C$$

According to $\lim\limits_{n\to\infty}\|\xi_{t_n} - \varrho_{t_n}\| = 0$, $\{\xi_{t_n}\}$, $\{\varrho_{t_n}\}$ are bounded, $\lambda > 0$, the limit of $\{\xi_{t_n}\}$ exists. We have

$$0 \leqslant \liminf_{n\to\infty}\langle F(\varrho_{t_n}), z - \varrho_{t_n}\rangle \leqslant \limsup_{n\to\infty}\langle F(\varrho_{t_n}), z - \varrho_{t_n}\rangle < +\infty$$

There are two Cases A and B below:

Case A: If $\limsup\limits_{n\to\infty}\langle F(\varrho_{t_n}), z - \varrho_{t_n}\rangle > 0$, this leads to the conclusion that there exists a subsequence $\{\varrho_{t_{n_i}}\}$ of $\{\varrho_{t_n}\}$ which exhibits the essential behavior that $\lim\limits_{i\to\infty}\langle F(\varrho_{t_{n_i}}), z - \varrho_{t_{n_i}}\rangle > 0$, $\exists N_0 \geqslant 0, \forall i \geqslant N_0$, $\langle F(\varrho_{t_{n_i}}), z - \varrho_{t_{n_i}}\rangle > 0$.

According to the quasimonotonicity of F, $\forall i \geqslant N_0, \langle F(z), z - \varrho_{t_{n_i}}\rangle \geqslant 0$, for $i \to \infty$, $\varrho_{t_{n_i}} \to \widehat{\xi}$, then $\langle F(z), z - \widehat{\xi}\rangle \geqslant 0, \forall z \in C$, so $\widehat{\xi} \in S_{\mathrm{D}}$.

Case B: If $\limsup\limits_{n\to\infty}\langle F(\varrho_{t_n}), z - \varrho_{t_n}\rangle = 0$, then

$$\lim_{n\to\infty}\langle F(\varrho_{t_n}), z - \varrho_{t_n}\rangle = \liminf_{n\to\infty}\langle F(\varrho_{t_n}), z - \varrho_{t_n}\rangle = \limsup_{n\to\infty}\langle F(\varrho_{t_n}), z - \varrho_{t_n}\rangle = 0$$

We set

$$\varepsilon_n = |\langle F(\varrho_{t_n}), z - \varrho_{t_n}\rangle| + \frac{1}{n+1}, \theta_{t_n} = \frac{F(\varrho_{t_n})}{\|F(\varrho_{t_n})\|^2}$$

Then, for $\forall n \in \mathbb{N}$, $\langle F(\varrho_{t_n}), z - \varrho_{t_n}\rangle + \varepsilon_n > 0$, $\langle F(\varrho_{t_n}), \theta_{t_n}\rangle = 1$.

We can get $\langle F(\varrho_{t_n}), z - \varrho_{t_n} + \varepsilon_n\theta_{t_n}\rangle > 0$, according to the quasimonotonicity of F, then $\langle F(z + \varepsilon_n\theta_{t_n}), z - \varrho_{t_n} + \varepsilon_n\theta_{t_n}\rangle \geqslant 0$. So

$$\begin{aligned}
&\langle F(z), z - \varrho_{t_n} + \varepsilon_n\theta_{t_n}\rangle = \\
&\quad \langle F(z) - F(z + \varepsilon_n\theta_{t_n}), z - \varrho_{t_n} + \varepsilon_n\theta_{t_n}\rangle + \\
&\quad \langle F(z + \varepsilon_n\theta_{t_n}), z - \varrho_{t_n} + \varepsilon_n\theta_{t_n}\rangle \geqslant \\
&\quad \langle F(z) - F(z + \varepsilon_n\theta_{t_n}), z - \varrho_{t_n} + \varepsilon_n\theta_{t_n}\rangle \geqslant \\
&\quad -\|F(z) - F(z + \varepsilon_n\theta_{t_n})\|\|z - \varrho_{t_n} + \varepsilon_n\theta_{t_n}\| \geqslant \\
&\quad -\varepsilon_n L\|\theta_{t_n}\|\|z - \varrho_{t_n} + \varepsilon_n\theta_{t_n}\| = \\
&\quad -\varepsilon_n\frac{L}{\|F(\varrho_{t_n})\|}\|z - \varrho_{t_n} + \varepsilon_n\theta_{t_n}\| \geqslant \\
&\quad -\varepsilon_n\frac{2L}{M_1}\|z - \varrho_{t_n} + \varepsilon_n\theta_{t_n}\|
\end{aligned}$$

We have $\|\theta_{t_n}\| = \dfrac{1}{\|F(\varrho_{t_n})\|}$, $\{\theta_{t_n}\}$ is bounded, since $\{\varrho_{t_n}\}$ is bounded, when $n \to \infty$, $\varepsilon_n \to 0$, so $\{\|z - \varrho_{t_n} + \varepsilon_n\theta_{t_n}\|\}$ is bounded. we have $\langle F(z), z - \widehat{\xi}\rangle \geqslant 0, \forall z \in C$, $\widehat{\xi} \in S_{\mathrm{D}}$.

Lemma 10 Produced by Algorithm 1, the resulting sequence $\{\xi_t\}$ possesses only finitely many cluster points within the set S.

Proof: We start by investigating whether the iterative process $\{\xi_t\}$ may converge weakly to several different limit points, rather than exhibiting uniqueness of its weak limit within the set S_{D}, under certain conditions. Suppose, contrary to what we aim to establish, that two distinct elements $\widehat{\xi} \in S_{\mathrm{D}}$ and $\bar{\xi} \in S_{\mathrm{D}}$ serve as a separate weak cluster of subsequences of $\{\xi_t\}$, with $\widehat{\xi} \neq \bar{\xi}$. Assume there exists a subsequence $\{\xi_{t_i}\}$ that converges weakly to some point $\bar{\xi}$ as i approaches infinity. Recalling that for $\lim\limits_{t\to\infty}\|\xi_t-\xi\|$ is well-defined for all $\xi\in S_{\mathrm{D}}$, we invoke Lemma 5 to proceed with the analysis,

$$\lim_{t\to\infty}\|\xi_t-\bar{\xi}\|=\lim_{i\to\infty}\|\xi_{t_i}-\bar{\xi}\|=\liminf_{i\to\infty}\|\xi_{t_i}-\bar{\xi}\|<\liminf_{i\to\infty}\|\xi_{t_i}-\widehat{\xi}\|=\lim_{t\to\infty}\|\xi_t-\widehat{\xi}\|=\lim_{n\to\infty}\|\xi_{t_n}-\widehat{\xi}\|=\liminf_{n\to\infty}\|\xi_{t_n}-\widehat{\xi}\|<\liminf_{n\to\infty}\|\xi_{t_n}-\bar{\xi}\|=\lim_{n\to\infty}\|\xi_{t_n}-\bar{\xi}\|=\lim_{t\to\infty}\|\xi_t-\bar{\xi}\|$$

This leads to a contradiction. Noting that the set $A=\{z\in C: F(z)=0\}\setminus S_{\mathrm{D}}$ contains only finitely many elements, by applying Lemma 9. This logical deduction leads to an important topological characteristic: the sequence $\{\xi_t\}$ possesses only finitely many weak cluster points within S.

Lemma 11 Suppose the set of all limit points of the iterative process $\{\xi_t\}$ is finite, denoted by $\xi_1, \xi_2, \xi_3, \cdots, \xi_p$. This structural property guarantees that there exists some sufficiently large index Z, all subsequent terms of the sequence will satisfy the condition that when $t>Z$, each ξ_t belongs to set $\mathcal{Q}$, where we define:

$$\mathcal{Q}=\bigcup_{j=1}^{p}\mathcal{Q}_j$$

$$\mathcal{Q}_l=\bigcap_{j=1,j\neq l}^{p}\left\{\xi:\langle\xi,\frac{\xi_l-\xi_j}{\|\xi_l-\xi_j\|}\rangle>\epsilon_0+\frac{\|\xi_l\|^2-\|\xi_j\|^2}{2\|\xi_l-\xi_j\|}\right\}$$

and the separation parameter ϵ_0 is given by

$$\epsilon_0=\min\left\{\frac{\|\xi_l-\xi_j\|}{4},l,j\in\{1,2,3,\cdots,p\},l\neq j\right\}$$

Proof: Consider a subsequence $\{\xi_{l_{t_i}}\}$ extracted from the original sequence $\{\xi_t\}$, which satisfies the condition that $\xi_{l_{t_i}}\rightharpoonup\xi_l$ as $i\to\infty$, we get $\forall j\neq l$,

$$\lim_{i\to\infty}\langle\xi_{l_{t_i}},\xi_l-\xi_j\rangle=\langle\xi_l,\xi_l-\xi_j\rangle \tag{17}$$

For $j\neq l$, we have

$$\langle\xi_l,\xi_l-\xi_j\rangle=\|\xi_l\|^2-\langle\xi_l,\xi_j\rangle=\frac{\|\xi_l-\xi_j\|^2}{2}+\frac{\|\xi_l\|^2}{2}-\frac{\|\xi_j\|^2}{2}>\frac{\|\xi_l-\xi_j\|^2}{4}+\frac{\|\xi_l\|^2}{2}-\frac{\|\xi_j\|^2}{2} \tag{18}$$

This implies that $\forall j\neq l$,

$$\langle\xi_l,\frac{\xi_l-\xi_j}{\|\xi_l-\xi_j\|}\rangle>\frac{\|\xi_l-\xi_j\|}{4}+\frac{\|\xi_l\|^2-\|\xi_j\|^2}{2\|\xi_l-\xi_j\|} \tag{19}$$

From Eq. (17) and inequality (19), for all i beyond a certain threshold, it follows that

$$\xi_{l_{t_i}}\in\left\{\xi:\langle\xi,\frac{\xi_l-\xi_j}{\|\xi_l-\xi_j\|}\rangle>\frac{\|\xi_l-\xi_j\|}{4}+\frac{\|\xi_l\|^2-\|\xi_j\|^2}{2\|\xi_l-\xi_j\|}\right\}$$

Therefore, when i is sufficiently large, we have $\xi_{l_{t_i}}\in\mathcal{Q}_l$, where

$$\mathcal{Q}_l=\bigcap_{j=1,j\neq l}^{p}\left\{\xi:\langle\xi,\frac{\xi_l-\xi_j}{\|\xi_l-\xi_j\|}\rangle>\epsilon_0+\frac{\|\xi_l\|^2-\|\xi_j\|^2}{2\|\xi_l-\xi_j\|}\right\} \tag{20}$$

and

$$\epsilon_0=\min\left\{\frac{\|\xi_l-\xi_j\|}{4}:l,j\in\{1,2,\cdots,p\},l\neq j\right\}$$

Next inclusion is immediate,

$$\mathcal{Q}_l=\bigcap_{j=1,j\neq l}^{p}\left\{\xi:\langle-\xi,\frac{\xi_l-\xi_j}{\|\xi_j-\xi_l\|}\rangle<-\epsilon_0+\frac{\|\xi_j\|^2-\|\xi_l\|^2}{2\|\xi_l-\xi_j\|}\right\} \tag{21}$$

Set $\mathcal{Q}=\bigcup\limits_{i=1}^{p}\mathcal{Q}_i$. The subsequent argument aims to demonstrate that this relation $\xi_t\in\mathcal{Q}$ eventually extends to the entire sequence for large enough t. Should this not be the case, one could identify a subsequence $\{\xi_{t_n}\}$ of $\{\xi_t\}$ for which $\xi_{t_n}\notin\mathcal{Q}(\forall n)$. The bounded nature of the sequence $\{\xi_{t_n}\}$ then guarantees the existence of a weakly convergent subsequence, hereafter simply referred to as $\{\xi_{t_n}\}\rightharpoonup\bar{\xi}\in C$. For convenience and without loss of generality, we continue to use the notation $\{\xi_{t_n}\}$ of the subsequence, which satisfies $\xi_{t_n}\rightharpoonup\bar{\xi}$. Noting that ξ_{t_n} does not belong to the set $\mathcal{Q}$, it follows that for $\forall l\in\{1,2,\cdots,p\}$, next certain condition holds,

$$\xi_{t_n}\notin\mathcal{Q}_l=\bigcap_{j=1,j\neq l}^{p}\left\{\xi:\langle\xi,\frac{\xi_l-\xi_j}{\|\xi_l-\xi_j\|}\rangle>\epsilon_0+\frac{\|\xi_l\|^2-\|\xi_j\|^2}{2\|\xi_l-\xi_j\|}\right\}$$

By invoking the principle of drawer, we can find a further subsequence $\{\xi_{t_{n_i}}\}$ extracted from $\{\xi_{t_n}\}$ and along with $l_0\in\{1,2,\cdots,p\}\setminus\{l\}$, such that the property in question remains valid for all $i\geqslant 0$,

$$\xi_{t_{n_i}}\notin\left\{\xi:\langle\xi,\frac{\xi_l-\xi_{l_0}}{\|\xi_l-\xi_{l_0}\|}\rangle>\epsilon_0+\frac{\|\xi_l\|^2-\|\xi_{l_0}\|^2}{2\|\xi_l-\xi_{l_0}\|}\right\}$$

We have

$$\bar{\xi}\notin\left\{\xi:\langle\xi,\frac{\xi_l-\xi_{l_0}}{\|\xi_l-\xi_{l_0}\|}\rangle>\epsilon_0+\frac{\|\xi_l\|^2-\|\xi_{l_0}\|^2}{2\|\xi_l-\xi_{l_0}\|}\right\} \tag{22}$$

Combining inequalities (18), (22) and

$$\langle\xi_l,\frac{\xi_l-\xi_{l_0}}{\|\xi_l-\xi_{l_0}\|}\rangle>\frac{\|\xi_l-\xi_{l_0}\|}{4}+\frac{\|\xi_l\|^2-\|\xi_{l_0}\|^2}{2\|\xi_l-\xi_{l_0}\|}\geqslant$$

$$\epsilon_0 + \frac{\|\xi_l\|^2 - \|\xi_{l_0}\|^2}{2\|\xi_l - \xi_{l_0}\|}$$

we get $\bar{\xi} \neq \xi_l$. Since l is arbitrary, we have $\bar{\xi} \notin \{\xi_1, \xi_2, \cdots \xi_p\}$, which leads to a contradiction. Therefore, the sequence elements ξ_t will lie in the set $\mathcal{Q}$ for all large enough t. In other words, there exists $Z_1 > Z$, such that for every $t \geqslant Z_1$, such that $\xi_t \in \mathcal{Q}$.

Theorem 1 A weak convergence result is established for Algorithm 1: the sequence $\{\xi_t\}$ of iterates that it produces is guaranteed to have its weak limit in the solution set S.

Proof: The conclusion of Lemma 8 provides the foundation for the next step, enabling us to deduce that $\lim\limits_{t\to\infty}\|\xi_{t+1} - \xi_t\| = 0$. It therefore follows that one can find an index $Z_2 > Z_1 > Z$, for every $t \geqslant Z_2$, the relation $\|\xi_{t+1} - \xi_t\| < \epsilon_0$ remains valid. We consider the scenario where the sequence $\{\xi_t\}$ admits multiple distinct weak cluster points. Then, applying Lemma 11, we deduce that there must be some $Z_3 \geqslant Z_2 > Z_1 > Z$ for which $\xi_{Z_3} \in \mathcal{Q}_l$, and $\xi_{Z_3+1} \in \mathcal{Q}_j$, where $l \neq j, l, j \in \{1, 2, \cdots, p\}$ and $p \geqslant 2$. Particularly, this ensures that:

$$\|\xi_{Z_3+1} - \xi_{Z_3}\| < \epsilon_0$$

Using inequalities (20) and (21), we get

$$\xi_{Z_3} \in \mathcal{Q}_l = \bigcap_{j=1, j\neq l}^{p} \left\{\xi : \left\langle \xi, \frac{\xi_l - \xi_j}{\|\xi_l - \xi_j\|}\right\rangle > \epsilon_0 + \frac{\|\xi_l\|^2 - \|\xi_j\|^2}{2\|\xi_l - \xi_j\|}\right\}$$

$$\xi_{Z_3+1} \in \mathcal{Q}_j = \bigcap_{l=1, l\neq j}^{p} \left\{\xi : \left\langle -\xi, \frac{\xi_j - \xi_l}{\|\xi_j - \xi_l\|}\right\rangle < -\epsilon_0 + \frac{\|\xi_l\|^2 - \|\xi_j\|^2}{2\|\xi_j - \xi_l\|}\right\}$$

Moreover, we obtain

$$\left\langle \xi_{Z_3}, \frac{\xi_l - \xi_j}{\|\xi_l - \xi_j\|}\right\rangle > \epsilon_0 + \frac{\|\xi_l\|^2 - \|\xi_j\|^2}{2\|\xi_l - \xi_j\|} \quad (23)$$

and

$$\left\langle -\xi_{Z_3+1}, \frac{\xi_l - \xi_j}{\|\xi_l - \xi_j\|}\right\rangle > \epsilon_0 + \frac{\|\xi_j\|^2 - \|\xi_l\|^2}{2\|\xi_l - \xi_j\|} \quad (24)$$

Combining inequalities (23) and (24), we get

$$2\epsilon_0 < \left\langle \xi_{Z_3} - \xi_{Z_3+1}, \frac{\xi_l - \xi_j}{\|\xi_l - \xi_j\|}\right\rangle \leqslant \|\xi_{Z_3+1} - \xi_{Z_3}\| < \epsilon_0 \quad (25)$$

The assumption leads to a contradiction. As a result, we must accept that the sequence $\{\xi_t\}$ possesses a single weak cluster point belonging to the set S. This observation leads directly to the deduction that $\xi_t \rightharpoonup \hat{\xi}$.

Theorem 2 Under the assumption that the operator F exhibits both δ-strongly pseudomonotone and L-Lipschitz continuous. Algorithm 1 produces an iterative sequence $\{\xi_t\}$ that exhibits strong convergence toward a unique solution within the set S.

Proof: From inequality (9), for $\hat{\xi} \in S$, we can get

$$\|\xi_{t+1} - \hat{\xi}\|^2 \leqslant \|\xi_t - \hat{\xi}\|^2 - \|\xi_t - \xi_{t+1}\|^2 + 2\lambda\langle F(\varrho_t), \hat{\xi} - \xi_{t+1}\rangle + 2\gamma_t\|\xi_{t+1} - \xi_t\|^2$$

From the property that the operator F is δ-strongly pseudomonotonic, we can get: for $\varrho_t \in C$, $\langle F(\hat{\xi}), \varrho_t - \hat{\xi}\rangle \geqslant 0$, then $\langle F(\varrho_t), \varrho_t - \hat{\xi}\rangle \geqslant \delta\|\varrho_t - \hat{\xi}\|^2$.

Therefore

$$\langle F(\varrho_t), \varrho_t - \xi_{t+1}\rangle = \langle F(\varrho_t), \hat{\xi} - \xi_{t+1}\rangle + \langle F(\varrho_t), \varrho_t - \hat{\xi}\rangle \geqslant \langle F(\varrho_t), \hat{\xi} - \xi_{t+1}\rangle + \delta\|\varrho_t - \hat{\xi}\|^2 \quad (26)$$

Putting inequality (26) into inequality (9), we can get

$$\|\xi_{t+1} - \hat{\xi}\|^2 \leqslant \|\xi_t - \hat{\xi}\|^2 - \|\xi_t - \xi_{t+1}\|^2 + 2\lambda\langle F(\varrho_t), \varrho_t - \xi_{t+1}\rangle + 2\gamma_t\|\xi_{t+1} - \xi_t\|^2 - 2\lambda\delta\|\varrho_t - \hat{\xi}\|^2$$

Rearranging the above inequality, we obtain

$$\|\xi_{t+1} - \hat{\xi}\|^2 \leqslant \|\xi_t - \hat{\xi}\|^2 - \|\xi_t - \varrho_k\|^2 - \|\varrho_t - \xi_{t+1}\|^2 + 2\gamma_t\|\xi_{t+1} - \xi_t\|^2 + 2\lambda\langle \xi_t - F(\varrho_t) - \varrho_t, \xi_{t+1} - \varrho_t\rangle - 2\lambda\delta\|\varrho_t - \hat{\xi}\|^2$$

Substituting Eq.(11) and inequality (12) into the above inequality, and utilizing L-Lipschitz continuity of F, we can get

$$\begin{aligned}\|\xi_{t+1} - \hat{\xi}\|^2 &\leqslant \|\xi_t - \hat{\xi}\|^2 - \|\xi_t - \varrho_t\|^2 - \|\varrho_t - \xi_{t+1}\|^2 + 2\gamma_t\|\xi_{t+1} - \xi_t\|^2 + 2\gamma_t\|\varrho_t - \xi_t\|^2 + 2\lambda\langle F(\xi_t) - F(\varrho_t), \xi_{t+1} - \varrho_t\rangle - 2\lambda\delta\|\varrho_t - \hat{\xi}\|^2 \\ &\leqslant \|\xi_t - \hat{\xi}\|^2 - \|\xi_t - \varrho_t\|^2 - \|\varrho_t - \xi_{t+1}\|^2 + 2\gamma_t\|\xi_{t+1} - \xi_t\|^2 + 2\gamma_t\|\varrho_t - \xi_t\|^2 + 2\lambda L\|\xi_t - \varrho_t\|\|\xi_{t+1} - \varrho_t\| - 2\lambda\delta\|\varrho_t - \hat{\xi}\|^2\end{aligned}$$

The next proof process is similar to inequalities (14), (15) and (16) of Lemma 8, we have

$$\|\xi_{t+1} - \hat{\xi}\|^2 \leqslant \|\xi_t - \hat{\xi}\|^2 - \eta_1\|\xi_t - \varrho_t\|^2 - \eta_2\|\xi_{t+1} - \varrho_t\|^2 - 2\lambda\delta\|\varrho_t - \hat{\xi}\|^2 +$$

$$\vartheta_1 \gamma_t \|F(\xi_t)\|^2$$

Since $\eta_1 > 0, \eta_2 > 0$,

$$\|\xi_{t+1} - \widehat{\xi}\|^2 \leqslant \|\xi_t - \widehat{\xi}\|^2 - 2\lambda\delta \|\varrho_t - \widehat{\xi}\|^2 + \vartheta_1 \gamma_t \|F(\xi_t)\|^2$$

Rearranging the above inequality, sum both sides simultaneously, we obtain

$$2\lambda\delta \sum_{t=1}^{\infty} \|\varrho_t - \widehat{\xi}\|^2 \leqslant \|\xi_1 - \widehat{\xi}\|^2 + \vartheta_1 \sum_{t=1}^{\infty} \gamma_t \|F(\xi_t)\|^2 < +\infty$$

Then, we have

$$\lim_{t\to\infty} \|\varrho_t - \widehat{\xi}\|^2 = 0 \tag{27}$$

Since $\|\xi_t - \widehat{\xi}\| \leqslant \|\xi_t - \varrho_t\| + \|\varrho_t - \widehat{\xi}\|$, therefore

$$\lim_{t\to\infty} \|\xi_t - \widehat{\xi}\|^2 = 0 \tag{28}$$

Remark 1 This study draws inspiration from a pre-existing iterative technique documented in Ref.[24], whose domain of applicability was previously confined to finite-dimensional Euclidean spaces and to problems exhibiting pseudomonotonicity. The central aim of the present paper is to transcend these limitations fundamentally. We develop a novel framework that not only lifts the dimensional constraint, allowing operation in infinite-dimensional Hilbert spaces, but also relaxes the operative assumptions, thus facilitating the solution of problems characterized by the broader and more challenging class of quasimonotone operators.

3 Numerical Experiment

Herein, we present a series of numerical experiments designed to demonstrate the practical performance and computational advantages of our method. All codes were written using PyCharm Community Edition 2024.1.1 and executed on a PC with an Intel(R) Core(TM) i7-7700T CPU @ 2.90 GHz and 8.00 GB of RAM.

To support the approximation of inexact projections without relying on exact calculations, the work in Ref. [31] introduces a widely recognized technique, the Frank-Wolfe (FW) algorithm. As described in Ref. [24], this approach seeks an approximate projection by solving a linear minimization problem over a closed convex set using a linear optimization oracle. The FW algorithm determines $\pi \in C$ serves as an admissible approximate projection of a given point $v \in H$ onto a convex and compact set C, which is defined in relation to a fixed reference point $u \in C$ and a nonnegative forcing parameter $\gamma \geqslant 0$, is given by the following formulation:

Initialization: Choose parameters $\gamma > 0$, initial points $v, u \in H$, and $\pi_0 \in C$. Set iteration counter $\mu := 0$.

Step 1: Use linear optimization oracle to compute:

$$\varphi_\mu := \operatorname{argmin}\{\langle \pi_\mu - v, y - \pi_\mu \rangle : y \in C\} \tag{29a}$$

$$\phi_\mu^* := \langle \pi_\mu - v, \varphi_\mu - \pi_\mu \rangle \tag{29b}$$

Step 2: Check stopping criterion:

If $-\varphi_\mu^* \leqslant \gamma \| \pi_\mu - u \|^2$, then stop and return

$$\pi := \pi_\mu \tag{30}$$

Step 3: Update iterate:

$$\alpha_\mu := \min\left\{1, \frac{-\phi_\mu^*}{\| \pi_\mu - \pi_\mu \|^2}\right\} \tag{31a}$$

$$\pi_{\mu+1} := \pi_\mu + \alpha_\mu(\varphi_\mu - \pi_\mu) \tag{31b}$$

Iteration: Set $\mu := \mu + 1$ and return to Step 1.

According to the analysis in Ref.[24], when the termination condition $-\phi_\mu^* \leqslant \gamma \| \pi_\mu - u \|^2$ is satisfied, the Frank-Wolfe procedure yields an approximate projection $\pi \in C$. Specifically, for any given $v \in H$ and reference point $u \in H$, the output satisfies the following inequality with respect to the error bound $\gamma \| \pi_\mu - u \|^2$: $\langle u - \pi, y - \pi \rangle \leqslant \gamma \| \pi_\mu - \pi \|^2, \forall y \in C$. It should be emphasized that the Frank-Wolfe algorithm introduced in our study is highly adaptable, capable of generating both approximate and precise projection results, by setting a sufficiently small tolerance, it is possible to calculate the exact projection. For further implementation details regarding the computation of approximate projections, see the comprehensive explanation in Ref.[32].

To illustrate the practical application of variational inequality problems, we analyze two distinct scenarios involving a pair of bounded linear operators.

Example 1 This article uses the examples in Ref.[4] for discussion. Let $C = [-1, 1]$ and

$$A\Theta = \begin{cases} 2\Theta - 1, & \Theta > 1 \\ \Theta^2, & \Theta \in [-1, 1] \\ -2\Theta - 1, & \Theta < -1 \end{cases}$$

Then, we have a mapping A that is quasimonotone and Lipschitz continuous. For this problem, we can also get $S_D = \{-1\}$ and $S = \{-1, 0\}$. The solution here is $\Theta^* = -1$.

The algorithm parameters are set as follows: $\varepsilon_t = (t+1)^{-3.1}$, $\bar{\gamma} \in [0.01, 0.5)$. The termination criterion is defined as $\|\xi_t - \varrho_t\| \leqslant 10^{-7}$. The inexact projection algorithm is used to solve the problem, and the outcomes of Algorithm 1 under different combinations of the parameters λ and $\bar{\gamma}$ are summarized in Table 1. The first column "Init. Pt." indicates the initial point, the third column "Iter" indicates how many iterations the extragradient method required to converge, while the fourth column "$\mathrm{Num}_{\mathrm{fw}}$" details the cumulative count of linear minimization steps executed by the Frank-Wolfe procedure, and the last column "CPU time (s)" represents the iteration time. Therefore, our algorithm can solve this quasimonotone variational inequality problem.

Table 1 Numerical performance with initial point $\xi_0 = -0.3$

Init. Pt.	λ	$\bar{\gamma}$	Iter	$\mathrm{Num}_{\mathrm{fw}}$	CPU time (s)
−0.3	0.12	0.01	19	12	0.0010
		0.11	19	12	0.0020
		0.21	19	12	0.0020
		0.31	19	12	0.0020
		0.41	19	12	0.0010
	0.27	0.01	8	8	0.0031
		0.11	8	8	0.0000
		0.21	8	8	0.0000
		0.31	8	8	0.0010
	0.42	0.01	6	12	0.0010
		0.11	6	12	0.0010

Example 2 The examples in Ref. [10] are used for discussion in this study. Consider the Hilbert space H, identified with the classical sequence space l_2, consisting of all infinite real sequences that meet a specific criterion described below:

$$\| g_1 \|^2 + \| g_2 \|^2 + \cdots + \| g_n \|^2 + \cdots < +\infty$$

Assume that $G: H \rightarrow H$ is defined by:

$$G(g) = (5 - \| g \|) g, \forall g \in H$$

where $C = \{ g \in H : \| g \| \leqslant 3 \}$. According to Ref. [10], we know that G is weakly continuous on C, and $\mathrm{VI}(G, C) = \{0\}$. Therefore, G is L-Lipschitz continuous, and $L = 11$. And the literature proves that G is quasimonotone on C.

$$\| G(g) - G(y) \| \leqslant 11 \| g - y \|$$

The algorithm parameters are set as follows: $\varepsilon_t = (t+1)^{-2.1}$, $\bar{\gamma} \in [0.01, 0.5)$, The stopping criterion is defined as when $\|\xi_t - \varrho_t\| \leqslant 10^{-7}$. The problem can be solved by using the inexact projection algorithm. The results obtained for different initial points and different combinations of the parameters λ and $\bar{\gamma}$ are shown in Tables 2 and 3. The first column "Init. Pt." indicates the initial point, the second column "Dim (m)" indicates the dimension of the initial point, the fifth column "Iter" indicates how many iterations the external gradient method requires to converge, and the last column "CPU time (s)" indicates the iteration time. Therefore, our algorithm can effectively solve this high-dimensional quasimonotone variational inequality problem.

Table 2 Numerical performance with initial point $\xi_0 = [1, 0, \cdots, 0_{9999}]$

Init. Pt.	Dim(m)	λ	$\bar{\gamma}$	Iter	CPU time (s)
$[1, 0, \cdots, 0_{9999}]$	10000	0.01	0.01	274	0.0636
			0.06	274	0.0558
			0.11	274	0.0758
			0.16	274	0.0776
			0.21	274	0.0814
			0.26	274	0.0678
		0.06	0.01	64	0.0150
			0.06	64	0.0160
			0.11	64	0.0519
			0.16	64	0.0120
			0.21	64	0.0190
			0.26	64	0.0199

Table 3 Numerical performance with initial point $\xi_0 = [500, \cdots, 500_{10000}, 0, \cdots, 0]$

Init. Pt.	Dim (m)	λ	$\bar{\gamma}$	Iter	CPU time(s)
$[500, \cdots, 500_{10000}, 0, \cdots, 0]$	20000	0.01	0.01	312	0.2563
			0.06	312	0.2500
			0.11	312	0.3820
			0.16	312	0.2334
			0.21	312	0.2434
			0.26	312	0.2653
		0.06	0.01	71	0.1666
			0.06	71	0.1606
			0.11	71	0.2932
			0.16	71	0.1636
			0.21	71	0.2882
			0.26	71	0.1705

4 Conclusions

This study extends an extragradient algorithm with inexact projection to solve quasi-monotone variational inequality problems in infinite-dimensional

Hilbert spaces. While preserving the projection-based iterative framework of the classical extragradient method, the proposed approach incorporates an inexact projection operator governed by a relative error criterion, which helps reduce computational burden without sacrificing feasibility. Under mild conditions, it is shown that if the operator is quasimonotone and Lipschitz continuous, the algorithm ensures weak convergence of the generated sequence. Furthermore, strong convergence is established when the operator satisfies δ -strong pseudomonotonicity on the set S over S_D . Numerical experiments confirm the effectiveness and practicality of the proposed method, demonstrating its potential for solving large-scale variational inequality problems.

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