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Regular and Singular Components of 2D Periodic Fluid Flows on a Surface of Viscous Stratified Fluid

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Abstract: The modern definition of the wave concept, which is based on the functional connection between the parameters of the spatial structure of an instantaneous flow pattern and the characteristics of the temporal variability at a given point, is discussed. The dispersion relation for 2D plane periodic perturbations on a surface of viscous stratified fluid is selected as a characteristic function defining the wave motion. Using the theory of singular perturbations, a method for calculating complete solutions to the dispersion relations of periodic flows, including regular wave and singular ligament solutions is presented. Properties of complete exact solution of the dispersion relation containing regular and singular functions are compared with asymptotic solutions. In limiting cases, obtained dispersion relations are matched with well-known expressions for waves in homogeneous viscous and ideal liquids.

Keywords: Navier-Stokes equation, periodic flows, theory of singular perturbations, asymptotic methods, surface capillary-gravity waves, ligaments, flow structure.

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0 Introduction

Waves have been observed on the surface of natural reservoirs, both small ponds and the boundless world ocean since time immemorial. However, it was only at the beginning of the 19th century that they became the subject of rigorous mathematical analysis, with the publication of Ref. [1]. The results of this work continued to be actively discussed and further developed in Ref. [2].

The observations of Franklin[3], who pointed out the importance of taking heterogeneity into account when studying fluid vibrations, were developed in works on the theoretical study of wave motions in a liquid layer of inhomogeneous density[4-5], in relation to the dynamics of an heterogeneous atmosphere[6-7]. The laboratory modelling of the "dead water" effect noticed by Nansen in Norwegian fjords and the Kara Sea was performed as well[8]. The results of the mathematical determination of buoyancy frequency, the frequency of natural fluctuations of displaced volume in stratified media, determined the frequency range for traveling internal waves[9], which received limited attention of researchers at the beginning of the 20th century. This loss led to an experimental redefinition of buoyancy frequency based on observations of probe ball movements[10] and atmospheric pressure microfluctuations[11]. However, the approximation of homogeneous medium, which was accepted in Ref. [1], was saved in subsequent studies on waves on the liquid surface performed in the 19th and the 20th centuries[12-13].

A large number of studies on traveling linear and nonlinear waves were based on the Euler equations[14]. The effect of internal friction was taken into account by introducing additional terms into the equations of motion[15-16]. It was partially considered in the theory of wave propagation by introducing a dissipative term (including kinematic viscosity) in the solutions of the ideal fluid equations of motion[9].

The interest in the detailed study of processes in the stratified ocean and atmosphere, which emerged in the second half of the 20th century, led to the

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expansion of the class of waves being investigated with different dispersion laws. This included not only surface waves, but also internal, inertial, and hybrid waves in both neutral and charged fluids[17-19]. In the 20th century, new methods for their mathematical description evolved. The important components of wave theory were the construction methods and the investigation of the properties of dispersion relations, which functionally related to the physically observable parameters of plane waves (period and wavelength), or their spectral counterparts (frequency and wave number) in linear and nonlinear approximations[20].

The results of wave observations in configuration space are normally described using real numbers. In order to derive the dispersion relations, the problem is submerged into the algebra of complex numbers, where both the frequency ω and wave vector $\boldsymbol{k}$ are typically chosen to be complex. This expansion of the mathematical formulation results in the appearance of more complicated mathematical quantities (the observed scalar wavelength correlates with a three-dimensional wave vector) and in the redundant solutions that are rejected based on physical criteria. Additionally, this approach leads to an increase in the number of methods available for analyzing the problem, including the calculations of flow stability[21].

In practice, a partial analysis of the problem is also carried out along with the general approach. The frequency indicating the energy is kept real, and the wave vector is chosen as complex number[22]. This means that the wavelength is determined by both the real and imaginary parts of the wave vector, and the wave attenuation is described by the imaginary component of the wave vector as it propagates. When studying waves in a weakly viscous liquid, such as water, where the system of governing equations contains small coefficients with higher derivatives, the methods of the theory of singular perturbations are used to find the solution to the system of equations[23-24]. Viscous stratified fluid flows have a large number of intrinsic eigenscale time and length scales, both in general case and in specific tasks. The ratios of the eigenscales define the small parameters of the problem, and the traditional dimensionless numbers, that are the Reynolds, Froude, Weber numbers, and others[25].

Modern researchers often resort to numerical methods when solving problems involving complex fluid flows, such as the flow of hybrid nanofluids over a rotating surface accounting for thermal radiation and electromagnetic field effects[26], flows in porous media[27], and magnetohydrodynamic flows of nanofluids[28]. (3) In Refs. [29]-[31], the influence of external factors such as magnetic field and medium porosity on enhancing heat transfer using nanofluids is investigated.

The modern definition of the wave concept is based on the functional connection between the parameters of the spatial structure of an instantaneous flow pattern and the characteristics of the temporal variability at a given point[22]. This connection is described by the dispersion relation. Its type depends on the physical parameters of the problem and the boundary conditions. The regular parts of the complete dispersion relations for periodic flows in a viscous continuously stratified fluid describe various types of waves (acoustic, gravitational, surface, internal and hybrid type). The regular components of the complete solution characterize waves that determine the geometry of the perturbed surface. Natural physical constraints define the acceptable direction of wave propagation.

The singular components define "ligaments"[25] describing the fine structure of the heterogeneous medium. They ubiquitously are manifested in the form of thin fibers and interfaces, which differ from a more homogeneous surrounding medium in their properties. The term "ligaments" was borrowed by Rayleigh[32] from medicine, where it refers to various types of connective tissues, such as "shells", "fascia", "fibers" and more than 30 other types, which are the synonyms of this term in this context.

The general approach allows us to take into account both the influence of gravity and the action of fields of a different physical nature. In particular, the effect of the surface electric charge on the characteristics of all components of periodic flows on the surface of a perturbed liquid is estimated in Ref. [25].

Given the prevalence of regular perturbations on the liquid surface and the scientific and particularly applied significance of studying them, it is important to describe the properties of all components of periodic flows and evaluate the completeness of the solutions obtained and the accuracy of the expressions received. This paper presents a methodology for creating and analyzing the properties of complete solution of the dispersion relation for periodic (or monochromatic) infinitesimal flows on the surface of a viscous, stably stratified liquid in a gravitational field. It also considers the effects of surface tension and compares the properties of the exact and approximate asymptotic solutions of the dispersion equation. The proposed methodology allows to obtain new complete solutions of the dispersion relations, including known regular formulae. The singular components of periodic flows describe ligaments determining the fine structure of periodic flows. The paper considers the application of the technique in relation to periodic surface disturbances in uniformly stratified viscous liquids with frequencies in the range of $0.01\ \mathrm{s}^{-1} < \omega < 1000\ \mathrm{s}^{-1}$.

1 Mathematical Formulation of the Problem

Due to the heterogeneity in the distribution of pressure, temperature, concentration of dissolved substances and suspended particles both liquids in natural reservoirs and the World Ocean and the atmosphere turn out to be stably stratified in the field of gravity. The stability of stratification, which is determined by the density distribution $\rho(z)$ with height z, and characterized by the length scale $\Lambda = |\mathrm{d}\ln\rho(z)/\mathrm{d}z|^{-1}$, the buoyancy frequency $N = \sqrt{g/\Lambda}$, and the period $T_b = 2\pi/N$ of buoyancy, appears to be heterogeneous. Relatively thick and more homogeneous layers with thin ligaments separated by thin highly gradient interfaces are distinguished in the profiles[33-34].

The complete system of equations for the mechanics of inhomogeneous fluids, which includes the continuity equations, the equations of transfer of momentum, total energy, and matter, as well as the closed equation of state with physically justified initial and boundary conditions[17-19, 22], is rather complex. That is why it would be advisable to start the development of methods for creating complete solutions, which describe infinitesimal periodic flows with the analysis of reduced models that enable the creation of complete solutions. One of these models is a system of motion equations for density-stratified fluids that do not contain additional physical variables describing fluid density distributions. This approach is often used to simplify calculations and reduce the number of variables when analyzing fluctuations and waves in liquids[17,35]. Additionally, it is assumed that the effects of compressibility can be neglected and an additional condition $\mathrm{div}\,\boldsymbol{u} = 0$ can be introduced for velocity $\boldsymbol{u}$.

The study of waves in heterogeneous fluids is carried out in the Cartesian coordinate system $Oxyz$. The Oz axis is directed vertically upward against the direction of gravity acceleration g. Perturbations of the free surface $z = \zeta(x,t)$ propagating in the positive direction of the horizontal axis Ox, which coincides with the equilibrium position of the free surface of the liquid, are considered.

The variable density of a liquid, which is used without specifying the physical nature of its change, is traditionally divided into two terms:

$$\rho(x,z,t) = \rho_{00}[r(z) + \tilde{\rho}(x,z,t)] = \rho_{00}[\exp(-z_\Lambda) + \tilde{\rho}(x,z,t)] \tag{1a}$$

$$z_\Lambda = z/\Lambda \tag{1b}$$

Here t denotes time variable, z_Λ is a dimensionless parameter characterizing the ratio of the current coordinate to the stratification scale Λ−the value of the depth at which the density increases by e times where "e" means Euler's number. The first term of Eq. 1 (a) describes an unperturbed density distribution with height, an exponential distribution $r(z) = \exp(-z_\Lambda)$ is chosen to simplify the calculations, and $\tilde{\rho}$ is a small density disturbance associated with the propagation of a periodic surface flow, ρ_{00} is the equilibrium density value at $z = 0$.

The pressure in a liquid $P = P(x,z,t)$ is written as the sum of atmospheric P_{00}, hydrostatic and periodic pressure perturbation $\tilde{P}(x,z,t)$:

$$P = P_{00} + \int_z^\zeta g\rho(x,\xi,t)\,\mathrm{d}\xi + \tilde{P}(x,z,t) \tag{2}$$

Here g denotes free fall acceleration, and ξ is the integration variable. Taking into account the simplifications, the mathematical formulation of the problem determines the dispersion characteristics, velocity field, momentum, and pressure in the fluid, which determines the dynamics and structure of periodic motion, consists of the Navier-Stokes and continuity equations, supplemented by physically justified boundary conditions on the free surface of the liquid, kinematic and dynamic boundary conditions:

$$z < \zeta: \begin{cases} \rho(\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) = \\ \quad \rho\nu\Delta\boldsymbol{u} - \nabla P + \rho\boldsymbol{g} \\ \partial_t\rho + \boldsymbol{u}\cdot\nabla\rho = 0 \\ \mathrm{div}\boldsymbol{u} = 0 \end{cases} \tag{3}$$

$$z = \zeta: \begin{cases} \partial_t(z - \zeta) + \boldsymbol{u} \cdot \nabla(z - \zeta) = 0 \\ \boldsymbol{\tau} \cdot ((\boldsymbol{n} \cdot \nabla)\boldsymbol{u}) + \boldsymbol{n} \cdot ((\boldsymbol{\tau} \cdot \nabla)\boldsymbol{u}) = 0 \\ P - P_0 - \sigma \operatorname{div}\boldsymbol{n} - 2\rho\nu n((\boldsymbol{n} \cdot \nabla)\boldsymbol{u}) = 0 \end{cases} \tag{4}$$

$$z = \to -\infty: \ \boldsymbol{u} = (u, w) \to (0,0) \tag{5}$$

$$\boldsymbol{n} = \frac{\nabla(z - \zeta)}{|\nabla(z - \zeta)|} = \frac{-\partial_x \zeta \boldsymbol{e}_x + \boldsymbol{e}_z}{\sqrt{1 + (\partial_x \zeta)^2}}$$

$$\boldsymbol{\tau} = \frac{\boldsymbol{e}_x + \partial_x \zeta \boldsymbol{e}_z}{\sqrt{1 + (\partial_x \zeta)^2}}$$

where $\boldsymbol{n}, \boldsymbol{\tau}$ are the vectors of the external normal and tangent to the free surface of the liquid, respectively, and $\boldsymbol{u} = (u, w)$ defines the velocity field in the liquid column caused by a surface disturbance.

Let us accept some conventional simplifications for this type of problem, namely the Boussinesq approximation. In the case of small amplitude deviations of the free surface, the Boussinesq approximation is applied when the density is assumed to be constant in terms with small coefficients. In this case, the density variability is taken into account only for terms containing a large coefficient (the gravity acceleration in the Navier-Stokes equation and the density gradient in the continuity equation). In the two-dimensional formulation, the velocity components can be expressed in terms of a single scalar stream function ψ:

$$\boldsymbol{u} = (u, w) = (\partial_z \psi, -\partial_x \psi) \tag{6}$$

Also, in the case of infinitesimal deviations, a well-known procedure, namely the domain perturbation method, is performed for the displacement of boundary conditions from wave surface to the equilibrium surface[36]. Then taking into account Eq.(6) and the accepted simplifications in a linear approximation, Eqs. (3)-(5) can be shown as follows:

$$z < 0: \begin{cases} \rho_{00} g \int_z^{\zeta} \partial_x \tilde{\rho}(x, \chi, t) \mathrm{d}\chi + \rho_{00} g \partial_x \zeta - \\ \quad \nu \rho_{00} \partial_z \Delta\psi + \rho_{00} \partial_{zt} \psi + \partial_x \tilde{P} = 0 \\ \nu \rho_{00} \partial_x \Delta\psi - \rho_{00} \partial_{xt} \psi + \partial_z \tilde{P} = 0 \\ \partial_t \tilde{\rho} + \dfrac{\exp(-z_\Lambda) \partial_x \psi}{\Lambda} = 0 \end{cases} \tag{7}$$

$$z = 0: \begin{cases} \partial_{zz}\psi - \partial_{xx}\psi = 0 \\ \tilde{P} + \sigma \partial_{xx}\zeta + 2\nu\rho \partial_{zx}\psi = 0 \\ \partial_t \zeta + \partial_x \psi = 0 \end{cases} \tag{8}$$

$$z = \to -\infty: \ (\partial_z \psi, -\partial_x \psi) \to (0,0) \tag{9}$$

Eqs. (7)-(9) are written for values of the first order of smallness of each unknown function. To maintain accuracy, overestimation is usually introduced and all unknown quantities in Eqs. (7)-(9) should additionally be marked with a subscript "1" indicating the order of magnitude, which is omitted to avoid cumbersome formula.

When performing mathematical calculations, it is often necessary to take into account the incompressibility condition. Besides the stream function, a hydrodynamic potential is introduced as well considering the motion to be potential. However, surface waves in stratified fluids lose their potential property due to baroclinic generation of vorticity[13]. The vorticity of these waves in stratified media is nonzero. When viscosity is regarded, additional terms appear in the mathematical formulation, which are responsible for generating vorticity in surface periodic flows[37]. Due to these observations, it becomes impossible to use velocity recordings through the hydrodynamic potential in the mathematical formulation of the given problem, as the solution quality is lost.

2 Solution of Linearized Problem

For surface periodic flows, the solution is sought in the form of harmonic functions:

$$f(x, z, t) \sim A\exp(\mathrm{i}k_x x + k_z z - \mathrm{i}\omega t) \tag{10}$$

Substituting Eq. (10) into Eq. (7), we obtain a dispersion relation connecting the components of the wave vector $\boldsymbol{k} = (k_x, k_z)$ with the frequency of wave motion ω:

$$\omega(k_x^2 - k_z^2)(\mathrm{i}\nu k_x^2 - \mathrm{i}\nu k_z^2 + \omega) - N^2 k_x^2 \exp(-z_\Lambda) = 0 \tag{11}$$

The proposed approach is based on the assumption that the frequency $\omega > 0$ is positive, as the parameter describes the energy of periodic flow. The components of $\boldsymbol{k} = (k_x, k_z)$ can be complex, and it can describe spatial attenuation of periodic disturbances.

It is convenient to analyze the dispersion relation Eq. (11) in a dimensionless form and to select eigenscale parameters as dimensionless values. The model easily identifies its spatial and temporal scales: viscous length scale $\delta_N^{g\nu} = (g\nu)^{1/3} N^{-1}$, buoyancy frequency N, buoyancy period T_b, stratification scale Λ, Stokes microscales for buoyancy $\delta_N^{\nu} = \sqrt{\nu/N}$ and wave $\delta_\nu^{\omega} = \sqrt{\omega/\nu}$ frequencies, and capillary length $\gamma_g^{\delta} = \sqrt{\gamma/g}$. The intrinsic scales determine the characteristic sizes and lifetimes of the flow components in a liquid, which can vary widely. For example, the spatial parameters of water can vary from fractions of centimeters to meters, both in laboratory and natural conditions. In this study, the values in Eq. (11) will become dimensionless. The inverse buoyancy frequency $\tau_N = N^{-1}$ will be the time scale and the viscous length scale $\delta_N^{g\nu} = \sqrt[3]{g\nu}/N$ will be the spatial scale. The dimensionless form of the dispersion relation Eq.(11) is written as:

$$\mathrm{i}\varepsilon\,(k_{*x}^2 - k_{*z}^2)^2\omega_* + (k_{*x}^2 - k_{*z}^2)\omega_*^2 - k_{*x}^2\exp(-z_\Lambda) = 0 \qquad (12)$$

where, the subscript "$*$" indicates the corresponding dimensionless values. With this choice of dimensionless parameters, the parameter $\varepsilon = \delta_g^\nu/\delta_N^{g\nu} = N\nu^{1/3}/g^{2/3}$ arises naturally in the problem defined by Eqs.(7)-(9), determined by the ratio of the natural scales of the medium. In a wide class of fluids, either the buoyancy frequency N or the kinematic viscosity ν, or both of these parameters are small quantities simultaneously, and therefore, ε is a natural small parameter in the problem defined by Eqs.(7)-(9).

To solve the dispersion relation Eq. (12) with respect to k_{*z}, the asymptotic methods are used[23-24]. The regular decomposition of the components of the wave vector k_{*z} with respect to the small parameter ε is written as:

$$k_{*z} = k_0 + \varepsilon k_1 + \varepsilon^2 k_2 + \ldots \qquad (13)$$

Substitute Eq.(13) into Eq.(12), we get:

$$\mathrm{i}\varepsilon\,(k_{*x}^2 - (k_0 + \varepsilon k_1 + \varepsilon^2 k_2 + \ldots)^2)^2\omega_* + (k_{*x}^2 - (k_0 + \varepsilon k_1 + \varepsilon^2 k_2 + \ldots)^2)\omega_*^2 - k_{*x}^2\exp(-z_\Lambda) = 0 \qquad (14)$$

By opening the brackets and collecting the multipliers with the same degree of the small parameter, we rewrite the dispersion relation as follows:

$$k_{*x}^2 - k_0^2\omega_* - \exp(-z_\Lambda) + \varepsilon(\mathrm{i}k_{*x}^4 - 2\mathrm{i}k_{*x}^2k_0^2\omega_* - 2k_0k_1\omega_*^2 + \mathrm{i}k_0^4\omega_*) + \varepsilon^2(-4\mathrm{i}k_{*x}^2k_0k_1\omega_* + 4\mathrm{i}k_0^3k_1\omega_* - k_1^2\omega_*^2 - 2k_0k_2\omega_*^2) + \cdots = 0 \qquad (15)$$

By allocating the terms corresponding to the degrees of ε, it is possible to construct a system of equations for different orders of magnitude:

$$\begin{cases} k_{*x}^2 - k_0^2\omega_* - \exp(-z_\Lambda) = 0 \\ \mathrm{i}k_{*x}^4 - 2\mathrm{i}k_{*x}^2k_0^2\omega_* - 2k_0k_1\omega_*^2 + \mathrm{i}k_0^4\omega_* = 0 \\ -4\mathrm{i}k_{*x}^2k_0k_1\omega_* + 4\mathrm{i}k_0^3k_1\omega_* - k_1^2\omega_*^2 - \\ \quad 2k_0k_2\omega_*^2 = 0 \\ \ldots \end{cases} \qquad (16)$$

By sequentially solving equations from Eq.(16), it is possible to obtain an approximate regular solution with any initially specified accuracy:

$$k_0 = \pm k_{*x}\frac{\sqrt{\omega_*^2 - \exp(-z_\Lambda)}}{\omega_*} \qquad (17\text{a})$$

$$k_1 = \frac{\mathrm{i}(k_{*x}^2 - k_0^2)}{2k_0\omega_*} = \pm\frac{\mathrm{i}k_*\exp(-z_\Lambda)}{2\omega_*^2\sqrt{\omega_*^2 - \exp(-z_\Lambda)}} \qquad (17\text{b})$$

$$k_2 = \frac{k_1(2\mathrm{i}k_0 - k_1\omega_*)}{2k_0\omega_*} = \frac{k_{*x}\exp(-z_\Lambda)(-3\exp(-z_\Lambda) + 4\omega_*^2)}{8\omega_*^2\sqrt{\omega_*^2 - \exp(-z_\Lambda)}(\omega_*^2 - \exp(-z_\Lambda))} \qquad (17\text{c})$$

Eq.(17) contains two roots, but Eq. (12) is a fourth-order relative to k_{*z}. Thus, using regular decompositions, we can only obtain a part of the solutions. It is worth be noting that in the dispersion Eq.(12), the small parameter ε stands at the highest power of k_{*z}. For clarity, expand the brackets in Eq. (12):

$$\mathrm{i}\varepsilon k_{*z}^4\,\omega_* + \mathrm{i}\varepsilon k_{*x}^4\omega_*^2 - 2\mathrm{i}\varepsilon k_{*x}^2k_{*z}^2\omega_* - k_z^2\omega_*^2 + k_{*x}^2\omega_*^2 - \exp(-z_\Lambda)k_{*x}^2 = 0 \quad (18)$$

The equation belongs to the singularly perturbed equations[24], and the missing solutions can be found using following singular expansions:

$$k_{*z} = \varepsilon^{-\eta}(k_0 + \varepsilon k_1 + \varepsilon^2 k_2 + \cdots),\ \eta > 0 \quad (19)$$

To determine the value of the parameter η, we put the main term of the expansion into the dispersion Eq.(18):

$$\mathrm{i}\varepsilon^{1-4\eta}\,k_0^4\omega_* + \mathrm{i}\varepsilon k_{*x}^4\omega_*^2 - 2\mathrm{i}\varepsilon^{1-2\eta}k_{*x}^2k_0^2\omega_* - \varepsilon^{-2\eta}k_0^2\omega_*^2 + k_{*x}^2\omega_*^2 - \mathrm{e}^{-z_\Lambda}k_{*x}^2 = 0 \qquad (20)$$

Following the procedure described in Ref.[24], we sequentially equate the degrees of the small parameter ε in different terms of the Eq.(20). As a result, the following relations arise for the variable η:

$$1 - 2\eta = 1, 1 - 2\eta = 1 - 4\eta, 1 - 2\eta = -2\eta, \quad 1 - 4\eta = 1, 1 - 4\eta = -2\eta, 1 - 4\eta = 0, \quad -2\eta = 1, -2\eta = 0, 1 - 2\eta = 0 \qquad (21)$$

With possible solutions:

$$\eta = -\frac{1}{2}, \eta = 0, \eta = \frac{1}{4}, \eta = \frac{1}{2} \qquad (22)$$

From the obtained roots, the value of η is selected based on the following considerations. When η from Eq. (22) is substituted into Eq. (20), it provides the main value for the term with the highest degree of k_0^4. In this case, the values of η, which is less than or equal to zero, are not considered since in this case decomposition will be reduced to the regular one discussed above. After substitution and verification, we find that one root $\eta = 1/2$ satisfies the specified conditions. The singular value decomposition is shown as follows:

$$k_{*z} = \varepsilon^{-1/2}k_0 + \varepsilon^{1/2}k_1 + \varepsilon^{3/2}k_2 + \ldots \tag{23}$$

Substituting Eq.(23) into Eq.(20), by analogy with the regular decomposition, we obtain the system of equations for terms of different orders of smallness in ε:

$$\begin{cases} k_0^2(k_0^2 + \mathrm{i}\omega_*) = 0 \\ \mathrm{i}\omega_*(2k_0k_1 - k_{*x}^2)(2k_0^2 + \mathrm{i}\omega_*) - \\ \quad \exp(-z_\Lambda)k_{*x}^2 = 0 \\ 6k_0^2k_1^2 + 4k_0^3k_2 - 4k_0k_1k_{*x}^2 + \\ \quad \mathrm{i}k_1^2\omega_* + 2\mathrm{i}k_0k_2\omega_* + k_{*x}^4 = 0 \\ \cdots \end{cases} \tag{24}$$

The sequential solution of this system allows to obtain asymptotic singular solutions:

$$k_0 = \pm\frac{1-\mathrm{i}}{\sqrt{2}}\sqrt{\omega_*} \tag{25a}$$

$$k_1 = k_*^2\frac{2\mathrm{i}k_0^2\omega_* - \omega_*^2 + \exp(-z_\Lambda)}{2k_0\omega_*(2\mathrm{i}k_0^2 - \omega_*)} = \pm\frac{(1+\mathrm{i})k_{*x}^2(\omega_*^2 - \exp(-z_\Lambda))}{2\sqrt{2}\omega_*^{5/2}} \tag{25b}$$

$$\cdots$$

It is worth noting that dispersion relation Eq.(12) allows constructing exact solutions in the model under consideration the following equtions:

$$k_{*z} = \pm\sqrt{k_{*x}^2 - \frac{\mathrm{i}\omega_*}{2\varepsilon} + \frac{\mathrm{i}\sqrt{4\mathrm{i}\varepsilon k_{*x}^2\exp(-z_\Lambda) + \omega_*^3}}{2\varepsilon\sqrt{\omega_*}}} \tag{26}$$

$$k_{*l} = \pm\sqrt{k_{*x}^2 - \frac{\mathrm{i}\omega_*}{2\varepsilon} - \frac{\mathrm{i}\sqrt{4\mathrm{i}\varepsilon k_{*x}^2\exp(-z_\Lambda) + \omega_*^3}}{2\varepsilon\sqrt{\omega_*}}} \tag{27}$$

Eqs.(26) – (27) correspond to roots in the following dimensional variables:

$$k_z = \pm\sqrt{k_x^2 - \frac{\mathrm{i}\omega}{2\nu} - \frac{(1-\mathrm{i})\sqrt{4k_x^2\nu\omega N^2\exp(-z_\Lambda) - \mathrm{i}\omega^4}}{2\sqrt{2}\nu\omega}} \tag{28}$$

$$k_l = \pm\sqrt{k_x^2 - \frac{\mathrm{i}\omega}{2\nu} + \frac{(1-\mathrm{i})\sqrt{4k_x^2\nu\omega N^2\exp(-z_\Lambda) - \mathrm{i}\omega^4}}{2\sqrt{2}\nu\omega}} \tag{29}$$

where, k_{*l} and k_l are redefined for roots that correspond to singular asymptotic solutions to distinguish between regular and singular solutions. The proof of compliance is given below.

To identify the correspondence between the exact solutions Eqs.(26) – (27) and the asymptotic solutions Eqs.(17) and (25), the root expression in Eqs.(26) – (27) in the Taylor series for the small parameter ε is as follows:

$$\frac{\mathrm{i}\sqrt{4\mathrm{i}\varepsilon k_{*x}^2\exp(-z_\Lambda) + \omega_*^3}}{2\varepsilon\sqrt{\omega_*}} \approx \frac{\mathrm{i}\omega_*}{2\varepsilon} - \frac{k_{*x}^2}{\omega_*^2}\exp(-z_\Lambda) \tag{30}$$

Substituting Eq.(30) into Eqs. (26) and (27), we get:

$$k_{*z} = \pm\sqrt{k_{*x}^2 - \frac{\mathrm{i}\omega_*}{2\varepsilon} + \frac{\mathrm{i}\omega_*}{2\varepsilon} - \frac{k_{*x}^2}{\omega_*^2}\exp(-z_\Lambda)} \approx \pm\frac{k_{*x}\sqrt{\omega_*^2 - \exp(-z_\Lambda)}}{\omega_*} \tag{31}$$

$$k_{*l} = \pm\sqrt{k_{*x}^2 - \frac{\mathrm{i}\omega_*}{2\varepsilon} - \left(\frac{\mathrm{i}\omega_*}{2\varepsilon} - \frac{k_{*x}^2}{\omega_*^2}\exp(-z_\Lambda)\right)} \approx \pm\frac{1-\mathrm{i}}{\sqrt{2\varepsilon}}\sqrt{\omega_*} \tag{32}$$

It should be noted that the approximate expansions obtained from Eqs.(31)–(32), exactly coincide with the first term of the asymptotic regular solution Eq. (24) and singular one Eq. (25), respectively. In this sense, the exact roots of Eqs. (26) and (28) can be called regular, and the roots of Eqs.(27) and (29) can be called singular.

Thus, qualitatively new solutions appear in the viscous fluid model, which are described by singular components of dispersion relations. They characterize the fine structure of periodic flows, i.e. ligaments. Taking into account the wave and ligament components of the flow Eqs.(26) – (27), the type of solution for the stream function ψ is written as follows:

$$\psi = (A\exp(k_z z) + B\exp(k_l z))\exp(\mathrm{i}k_x x - \mathrm{i}\omega t) \tag{33}$$

From the compatibility of the boundary conditions Eq.(8) and the basic Eq. (7), we obtain a dispersion equation connecting the component of the wave vector $\boldsymbol{k}_x$ with the frequency ω. To do this, we exclude the free surface deviation function of the equilibrium value ζ from the equations using the cross derivative of the kinematic and dynamic boundary conditions. The periodic pressure perturbation $\tilde{P}(x,z,t)$ is excluded due to the implication of the component of the Navier-Stokes Eq. (7), and by substituting

Eq.(33), we obtain the following ratio[38]:

$$(k_x^2 + k_z^2)(k_l\omega^2 - gk_x^2 - \gamma k_x^4 + \mathrm{i}\omega\nu k_l(3k_x^2 - k_l^2)) - (k_x^2 + k_l^2)\cdot (k_z\omega^2 - gk_x^2 - \gamma k_x^4 + \mathrm{i}\omega\nu k_z \cdot (3k_x^2 - k_z^2)) = 0 \quad (34)$$

The terms determined by gravity (proportional to $\propto \sim gk_x^2$) and the terms characterizing the effect of surface tension (proportional to γk_x^4) are easily distinguished in the dispersion relation Eq. (34). Gravitational and capillary effects are often considered separately in practical problems. The dispersion equation becomes significantly simpler with these wavelength constraints. When both effects are taken into account, the dispersion Eq. (34) is not solved exactly analytically, but can be solved numerically using modern programs for symbolic calculations. The use of asymptotic methods allows obtaining approximate solutions. The remaining boundary conditions allow us to determine the relationship between the amplitudes of the ligament and wave components of the stream function and the deviation of the free surface from the equilibrium position. In a dimensionless form, when ratios of the eigenscales are used as the dimensionless parameters, the ratio Eq.(34) is rewritten as follows:

$$(k_{*x}^2 + k_{*l}^2)(\delta^2\varepsilon k_{*x}^4 + \mathrm{i}\varepsilon^2\omega_* k_{*z}(k_{*z}^2 - 3k_{*x}^2) + k_{*x}^2 - \varepsilon k_{*z}\omega_*^2) - (k_{*x}^2 + k_{*z}^2)(\delta^2\varepsilon k_{*x}^4 + \mathrm{i}\varepsilon^2\omega_* k_{*l}(k_{*l}^2 - 3k_{*x}^2) + k_{*x}^2 - \varepsilon k_{*l}\omega_*^2) = 0 \quad (35)$$

where $\delta = \delta_g^\gamma/\delta_N^\nu = \sqrt{N\gamma/\nu g}$ is a dimensionless parameter characterizing the ratio of the capillary length to the Stokes microscale. Substituting Eqs.(31) and (32) into Eq. (35), we can obtain an approximate dispersion relation up to terms of the order of $o(\varepsilon)$:

$$k_{*x}\left(\frac{1-\mathrm{i}}{\sqrt{2\varepsilon}}\sqrt{\omega_*} - k_{*x}\frac{\sqrt{\omega_*^2 - 1}}{\omega_*}\right)\cdot \left[\frac{1-i}{\sqrt{2\varepsilon}}\sqrt{\omega_*}k_{*x} + \frac{\sqrt{\omega_*^2-1}}{\omega_*}k_{*x}^2 + \left(\frac{1-\mathrm{i}}{\sqrt{2}}\delta^2\sqrt{\omega_*}k_{*x}^3 - \frac{1-\mathrm{i}}{\sqrt{2}}\omega_*^{3/2}\sqrt{\omega_*^2-1}\right)\sqrt{\varepsilon} + \left(k_{*x}(1-\omega_*^2) + \frac{\delta^2\sqrt{\omega_*^2-1}}{\omega_*}k_{*x}^4\right)\varepsilon\right] = 0 \quad (36)$$

Reducing the approximate dispersion relation Eq. (36) to a dimensional form and simplifying the expression, we obtain:

$$k_x(-\mathrm{i}\omega^3 + k_x^2\nu(N^2-\omega^2))\cdot (k_x^3\gamma + gk_x - \omega\sqrt{\omega^2 - N^2}) = 0 \quad (37)$$

The roots of Eq.(37) are easy to find and take values:

$$k_x = 0,\ k_x = \pm\frac{(1+\mathrm{i})\omega^{3/2}}{\sqrt{2\nu(N^2-\omega^2)}} \quad (38)$$

$$k_x = \frac{\alpha^{1/3}}{3^{2/3}2^{1/3}\gamma} - \frac{2^{1/3}g}{3^{1/3}\alpha^{1/3}} \quad (39)$$

$$k_x = -\frac{(1\mp\sqrt{3}\mathrm{i})\alpha^{1/3}}{2^{4/3}3^{2/3}\gamma} + \frac{(1\pm\sqrt{3}\mathrm{i})g}{2^{2/3}3^{1/3}\alpha^{1/3}} \quad (40)$$

$$\alpha = 9\gamma^2\omega\sqrt{\omega^2-N^2} + \sqrt{3\gamma^3(4g^3 + 27\gamma\omega^2(\omega^2-N^2))}$$

The solutions of the dispersion equations must be provided with natural conditions for the physical realization of the roots. For surface periodic two-dimensional flows propagating in the positive direction of the Ox axis, the conditions for the physical feasibility of solutions are written as follows:

$$\mathrm{Re}(k_x) > 0,\mathrm{Im}(k_x) > 0,\mathrm{Re}(k_{z,l}) > 0 \quad (41)$$

The positivity condition of the imaginary part of the component k_x is satisfied by Eqs. (39) and (38). The remaining roots grow as the horizontal coordinate increases. When we substitute these expressions into Eqs. (26) − (27), we find that only the solutions defined by the root of Eq. (39) exhibit a decrease with decreasing vertical coordinate.

To compare the solutions obtained using asymptotic methods with those from the numerical calculation of the dispersion equation, we construct the dispersion dependencies for the wave and segment components of the surface periodic flow in a fluid with water parameters $\sigma = 72\times10^{-7}\,\mathrm{J/cm^2}$, $\rho_{00} = 1\ \mathrm{g/cm^3}$, $g = 981\ \mathrm{cm/s^2}$ for laboratory conditions (highly stratified liquid $N = 1\ \mathrm{s^{-1}}$) and for a stratified ocean ($N=0.01\ \mathrm{s^{-1}}$). We define the wavelength λ and the scale of the ligament δ_l as follows[38]:

$$\lambda = \frac{2\pi}{\sqrt{\mathrm{Re}\,(k_x)^2 + \mathrm{Im}\,(k_z)^2}}$$

$$\delta_l = \frac{2\pi}{\sqrt{\mathrm{Re}\,(k_x)^2 + \mathrm{Im}\,(k_l)^2}}$$

The curves in Figs.1(a) and 1(b) are performed using the expressions (26) and analytical asymptotic Eq. (39) for analytical wavelength $\lambda_a(\omega)$ (solid lines) and obtained as a result of the direct numerical solution of the dispersion Eq. (34) for numerical wavelength $\lambda_n(\omega)$ (dashed lines). Similar graphics are performed for analytical ligament scale $\delta_{la}(\omega)$ Eqs. (27) and (39) (solid lines) and obtained as a result of the direct numerical solution of the dispersion Eq. (34) for numerical ligament scale $\delta_{ln}(\omega)$ (dashed lines).

In Figs. 1(a) and (b), the dependencies of the analytical asymptotic wavelength λ_a and numerical wavelength λ_n on the frequency of the periodic flow with insets of the region are constructed. The difference for surface periodic flows is most noticeable in a liquid where water parameters have different values of buoyancy frequency. In Figs. 1 (c) and (d), the difference relative to the wavelength of λ_n between the analytical and numerical values of $\Delta\lambda/\lambda = |\lambda_a - \lambda_n|/\lambda_n$ for different values of the buoyancy frequency is shown.

The analysis shows that the most noticeable difference between asymptotic and numerical values is achieved in the field of capillary-gravity waves. It takes values of the order of percent, starting from the frequency of 20 s^{-1}, and grows with the increasing frequency up to values of the order of several tens of percent. As the frequency of periodic movement decreases, the relative difference decreases rapidly and amounts to fractions of a percent. As the value of the periodic motion frequency approaches the buoyancy frequency, there is an increasing difference between the numerical and analytical asymptotic solution, which is more noticeable for liquids with lower buoyancy frequency values and does not exceed fractions of a percent. For a strongly stratified liquid, the analytical asymptotic value underestimates the wavelength compared with the numerical solution, while for a weakly stratified liquid, on the contrary, an overestimation is observed. This is due to the presence of the buoyancy frequency in the small parameter ε and the discarding of some terms in asymptotic calculations.

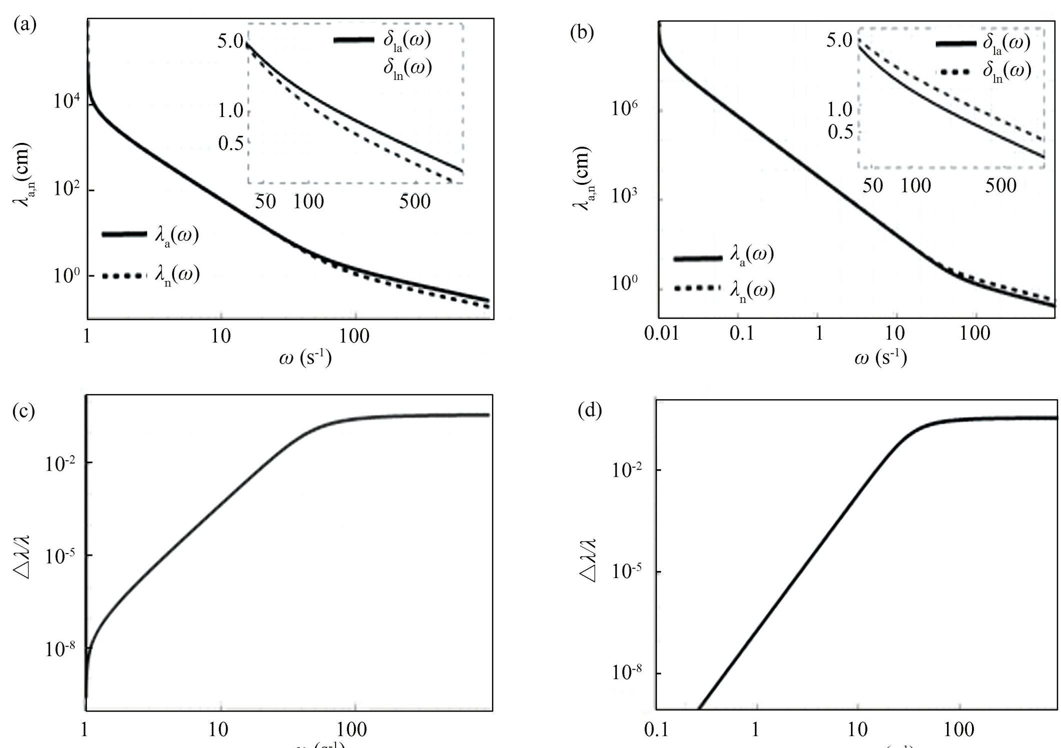

Fig. 1 The dependence of the wavelength on the frequency of periodic motion obtained using analytical asymptotic methods λ_a and as a result of the numerical solution λ_n at: (a) $N=1$ s^{-1}, (b) $N=0.01$ s^{-1} and the relative difference between numerical and analytical asymptotic expressions $\Delta\lambda/\lambda_n$ at (c) $N=1$ s^{-1}, (d) $N=0.01$ s^{-1}

In Fig. 2, which is similar to Fig. 1, the dependencies for the ligament component of the flow are shown. Figs. 2 (a) and (b) show the dependencies of the analytical δ_{la} and numerical δ_{ln}

ligament scale on frequency at different values of the buoyancy frequency. In Figs. 2 (c) and (d), the difference between the asymptotic and numerical values of $\Delta\delta_l/\delta_l = |\delta_{la} - \delta_{ln}|/\delta_{ln}$ relative to the numerical scale of the ligament δ_{ln} is created. When calculating fine-structured ligaments, the difference is insignificant, and the analytical asymptotic expressions practically coincide with the numerical ones for the entire frequency range.

3 Limiting Transitions

Considering the transitions to the known limiting cases and observing the changes in the regular and singular components of the solution.

3.1 Homogeneous Viscous Fluid

The most common simplification in the theory of surface waves is the investigation of the problem without taking into account density inhomogeneity, while considering the influence of viscosity. In a homogeneous liquid $\rho = \rho_{00} = \text{const}$, Eqs. (2) - (5) becomes significantly simpler. The pressure distribution in a liquid can be determined by the following expression:

$$P = P_{00} + \rho_{00} g(\zeta - z) + \tilde{P}(x,z,t)$$

The equation of continuity is automatically satisfied under the given simplifications. The basic equations of motion and the boundary conditions can be expressed as:

$$z < \zeta: \begin{cases} \rho_{00}(\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) = \\ \quad \rho_{00}\nu\Delta\boldsymbol{u} - \nabla P + \rho_{00}\boldsymbol{g} \\ \text{div}\boldsymbol{u} = 0 \end{cases}$$

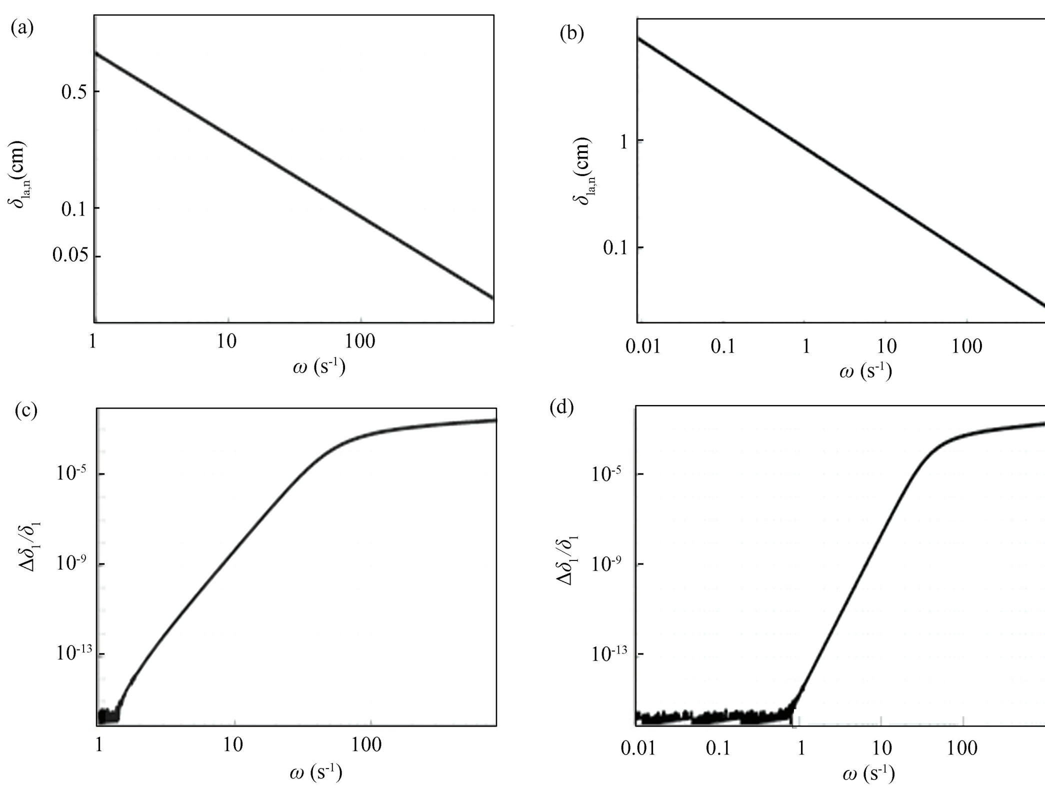

Fig. 2 The dependence of the ligament scale on the frequency of periodic motion obtained using analytical asymptotic methods δ_{la} and as a result of the numerical solution δ_{ln} at: (a) N=1 s^{-1}, (b) N = 0.01 s^{-1} and the relative difference between numerical and analytical asymptotic expressions $\Delta\delta_l/\delta_l$ at (c) N=1 s^{-1}, (d) N=0.01 s^{-1}

$$z = \zeta: \begin{cases} \partial_t(z-\zeta) + \boldsymbol{u}\cdot\nabla(z-\zeta) = 0 \\ \boldsymbol{\tau}\cdot((\boldsymbol{n}\cdot\nabla)\boldsymbol{u}) + \boldsymbol{n}\cdot((\boldsymbol{\tau}\cdot\nabla)\boldsymbol{u}) = 0 \\ P - P_0 - \sigma\,\text{div}\boldsymbol{n} - \\ \quad 2\rho_{00}\nu\boldsymbol{n}((\boldsymbol{n}\cdot\nabla)\boldsymbol{u}) = 0 \end{cases}$$

$$z = \to -\infty: \ \boldsymbol{u} = (u,w) \to (0,0)$$

The expressions for the stream function take the form:

$$z < 0: \begin{cases} \rho_{00} g\partial_x\zeta - \nu\rho_{00}\partial_z\Delta\psi + \rho_{00}\partial_{zt}\psi + \partial_x\tilde{P} = 0 \\ \nu\rho_{00}\partial_x\Delta\psi - \rho_{00}\partial_{xt}\psi + \partial_z\tilde{P} = 0 \end{cases} \tag{42}$$

$$z = 0: \begin{cases} \partial_{zz}\psi - \partial_{xx}\psi = 0 \\ \tilde{P} + \sigma\partial_{xx}\zeta + 2\nu\rho_{00}\partial_{zx}\psi = 0 \\ \partial_t\zeta + \partial_x\psi = 0 \end{cases} \qquad (43)$$

$$z = \to -\infty: (\partial_z\psi, -\partial_x\psi) \to (0,0)$$

By substituting Eq. (10) into the system (42), we can obtain the dispersion relation that connects the components of the wave vector:

$$(k_x^2 - k_z^2)(\mathrm{i}\nu k_x^2 - \mathrm{i}\nu k_z^2 + \omega) = 0 \qquad (44)$$

Solving Eq.(44), we get the following roots:

$$k_z = \pm k_x, k_l = \pm\sqrt{k_x^2 - \frac{\mathrm{i}\omega}{\nu}} \qquad (45)$$

Here the solution obtained using the theory of singular perturbations is reinterpreted (k_l) by analogy with the previous item. Considering the roots of Eq.(45), the solution for the stream function takes the form of Eq. (33) by analogy with the previous item. The dispersion relation comes from the compatibility condition between the boundary conditions and the basic Eqs.(42)-(43). It provides the connection between the components of the wave vector, frequency, and other parameters in the problem. This formally looks identical to Eq. (34).

$$\begin{aligned} &(k_x^2 + k_z^2)(k_l\omega^2 - gk_x^2 - \gamma k_x^4 + \\ &\mathrm{i}\omega\nu k_l(3k_x^2 - k_l^2)) - (k_x^2 + k_l^2)\cdot \\ &(k_z\omega^2 - gk_x^2 - \gamma k_x^4 + \\ &\mathrm{i}\omega\nu k_z(3k_x^2 - k_z^2)) = 0 \end{aligned} \qquad (46)$$

The solution of Eq. (46) can be found by taking into account Eq. (45). It can be obtained either numerically or using analytical asymptotic methods. More details on the peculiarities of calculating dispersion relations in simplified models are discussed in Ref.[38].

Dispersion relations can be derived by performing the limiting transition $N \to 0$ ($\Lambda \to \infty$). The regular components of the flow determine large-scale waves as in a stratified fluid, while the singular components correspond to fine ligaments. However, the density of the medium is not considered in a homogeneous fluid model, therefore the density gradient is equal to zero and it would not be correct to speak of ligament structures as high-gradient interfaces. Hence, the quality of singular components of the solution is lost in a homogeneous fluid model.

3.2 Ideal Stratified Fluid

When considering flows in the model of an ideal stratified fluid, the mathematical formulation of the problem can be written as:

$$z < \zeta: \begin{cases} \rho(\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) = -\nabla P + \rho\boldsymbol{g} \\ \partial_t\rho + \boldsymbol{u}\cdot\nabla\rho = 0 \\ \mathrm{div}\boldsymbol{u} = 0 \end{cases}$$

$$z = \zeta: \begin{cases} \partial_t(z - \zeta) + \boldsymbol{u}\cdot\nabla(z - \zeta) = 0 \\ P - P_0 - \sigma\,\mathrm{div}\boldsymbol{n} = 0 \end{cases}$$

$$z = \to -\infty: \boldsymbol{u} = (u, w) \to (0,0)$$

The mathematical formulation of the problem takes the following form in a linear approximation when the stream function is applied:

$$z < 0: \begin{cases} \rho_{00}g\int_z^\zeta \partial_x\tilde{\rho}(x,\chi,t)\,\mathrm{d}\chi + \\ \quad \rho_{00}g\partial_x\zeta + \rho_{00}\partial_{zt}\psi + \partial_x\tilde{P} = 0 \\ -\rho_{00}\partial_{xt}\psi + \partial_z\tilde{P} = 0 \\ \partial_t\tilde{\rho} + \dfrac{\exp(-z_\Lambda)\partial_x\psi}{\Lambda} = 0 \end{cases} \qquad (47)$$

$$z = 0: \begin{cases} \tilde{P} + \sigma\partial_{xx}\zeta = 0 \\ \partial_t\zeta + \partial_x\psi = 0 \end{cases}$$

$$z = \to -\infty: (\partial_z\psi, -\partial_x\psi) \to (0,0)$$

The substitution of Eq. (10) into the system (47) in an ideal stratified fluid results in the following dispersion equation:

$$(k_x^2 - k_z^2)\omega^2 - N^2k_x^2\exp(-z_\Lambda) = 0 \qquad (48)$$

The solution of Eq. (48) is written as:

$$k_z = \pm k_x\sqrt{1 - \frac{N^2}{\omega^2}\exp(-z_\Lambda)} \qquad (49)$$

The classical relation that defines the connection between the components of the wave vector and the frequency of the wave is derived from the compatibility of the boundary conditions on the free surface and the basic equations of motion:

$$gk_x^2 - \omega^2k_z + \gamma k_x^4 = 0$$

The rate of the system of equations describing the motion is reduced due to the disregard of viscosity in the basic equations. This results in a decrease of the order of dispersion equation, and the loss of some solutions, i.e. there are no singular roots in the ideal fluid model. Therefore, a whole class of solutions that describe structural components is lost in the problems involving the propagation of periodic flows due to the disregard of viscosity.

3.3 Ideal Homogeneous Fluid

The mathematical description in the ultimately reduced model of an ideal homogeneous fluid is expressed in the simplest terms as follows:

$$z < \zeta: \begin{cases} \rho_{00}(\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}) = -\nabla P + \rho_{00}\boldsymbol{g} \\ \mathrm{div}\boldsymbol{u} = 0 \end{cases}$$

$$z = \zeta: \begin{cases} \partial_t(z-\zeta) + \boldsymbol{u} \cdot \nabla(z-\zeta) = 0 \\ P - P_0 - \sigma \mathrm{div}\boldsymbol{n} = 0 \end{cases}$$

$$z = \to -\infty: \boldsymbol{u} = (u, w) \to (0,0)$$

Using the stream function in a linear approximation, the mathematical formulation of the problem can be written as follows:

$$z < 0: \begin{cases} \rho_{00} g \partial_x \zeta + \rho_{00}\partial_{zt}\psi + \partial_x \tilde{P} = 0 \\ -\rho_{00}\partial_{xt}\psi + \partial_z \tilde{P} = 0 \end{cases}$$

$$z = 0: \begin{cases} \tilde{P} + \sigma \partial_{xx}\zeta = 0 \\ \partial_t \zeta + \partial_x \psi = 0 \end{cases}$$

$$z = \to -\infty: (\partial_z \psi, -\partial_x \psi) \to (0,0)$$

The dispersion relations in the model of an ideal homogeneous fluid are greatly simplified and include only the wave components of the periodic flow. Thus, the surface periodic flow is found to be potential in this model. Analogous expressions can be derived using the velocity potential $\boldsymbol{u} = \nabla\varphi$.

$$k_z = \pm k_x, g k_x^2 - \omega^2 k_z + \gamma k_x^4 = 0 \qquad (50)$$

It is worth noting that the choice of the sign in Eqs. (45), (49), and (50) is made in accordance with the conditions for the physical realization of the roots Eq. (41), for the reasons similar to those discussed in Section 2.

A highly simplified model leads to a straightforward solution that captures some trends in large-scale flow components, but it differs qualitatively from the solutions obtained in more complete formulations. The key difference is the absence of certain solutions that define the fine structure of periodic processes in continuous media.

4 Conclusions

The investigation of the linearized equations of motion for a heterogeneous viscous fluid shows the complexity of the dispersion relation, which forms the basis for distinguishing the wave motion from other types of flows. The high rank of the system and the order of the resulting dispersion relation require the simplification of the problem and the implication of approximate analysis methods. We use the Boussinesq approximation to simplify the expressions. The variable density with constant buoyancy frequency remains only in terms with large coefficients in our calculations. The assumption of low viscosity allows us to apply the theory of singular perturbations. The complete solution to the dispersion equation includes regular terms that characterize waves and singular terms that describe ligaments. Comparison of the precise and asymptotic solutions reveals their good agreement. Neglesting the heterogeneity and viscosity of the medium, we obtain known solutions. At the same time, the quality of the reduced expressions is decreased as far as in a homogeneous medium without dissipation the whole class of singular solutions, and the mathematical description of ligaments are lost. Constructed complete solutions of linearized periodic motion problem in heterogeneous media can be used to solve the problems of wave generation and other components of flows with physically justified initial and boundary conditions.

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