

**Citation:** Hao Chen, Xiaolong Xin. Triangle bounded  $L$ -algebras and triangle ideals. *Journal of Harbin Institute of Technology (New Series)*. DOI: 10.11916/j.issn.1005-9113.2024059

# Triangle Bounded $L$ -algebras and Triangle Ideals

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**Abstract:** This study mainly focuses on the triangle bounded  $L$ -algebras and triangle ideals. Firstly, the definition of triangle bounded  $L$ -algebras is presented, and several examples with different conditions are outlined along with an exploration of their properties. Moreover, we investigate the structure of triangle bounded  $L$ -algebra with special condition. Secondly, we define the concept of triangle ideals of triangle bounded  $L$ -algebra and explore the connection between the triangle ideals of triangle bounded  $L$ -algebra  $L$  and the ideals of bounded  $L$ -algebra  $E(L)$ . In addition, we classified and studied various classes of triangle ideals, including Stonean triangle ideals, extended Stonean triangle ideals, and lattice ideals, and by introducing the notion of Stonean triangle bounded  $L$  algebras, we examine the relationship between Stonean triangle bounded  $L$ -algebras and Stonean triangle ideals. Finally, we investigate the interrelationships among these various types of triangle ideals.

**Keywords:** triangle bounded  $L$ -algebra; Stonean triangle bounded  $L$ -algebra; triangle ideal; (extended) Stonean triangle ideal; lattice ideal

**CLC number:** O29

**Document code:** A

**Article ID:** 1005-9113(2025)00-0000-13

## 0 Introduction

Non-classical logic turns out to be a valuable device for computers to handle uncertain and fuzzy information.  $L$ -algebra has the characteristics of multivalued logic, intuitionistic logic, and quantum logic, therefore it is closely related to algebraic logic and quantum structure, and is a non-classical logical algebra worth studying. In 2008, Rump<sup>[1]</sup> introduced and investigated the  $L$ -algebras, exploring their characteristics and properties. Additionally, various special definitions of  $L$ -algebras were presented, and it was demonstrated that every  $L$ -algebra admits a self-similar closure. In 2012, Rump and Yang<sup>[2]</sup> provided an equivalent characterization of intervals in lattice ordered groups that can be represented by  $L$ -algebra. In 2019, Wu et al.<sup>[3]</sup> explored its relationship with  $L$ -algebra by constructing effect algebra. In 2021, Ciungu<sup>[4]</sup> provided characterizations of a special  $L$ -algebra and explores its relationship with other algebraic structures. In 2023, Kologani<sup>[5]</sup> investigated the relationship between the quotient structure of

$CKL$ -algebras and  $BCK$ -algebras. In 2024, Hu et al.<sup>[6]</sup> studied bounded algebras from a different perspective by using a new kind of subalgebra (dual ideals) in bounded  $L$ -algebras. In the same year, Yun et al.<sup>[7]</sup> studied different ideals of  $CKL$ -algebras. Consequently,  $L$ -algebras have turned into a focal point of logic algebra study over the past decade.

When studying algebraic structures, triangle algebras are a powerful tool. In 2008, Van Gasse et al.<sup>[8]</sup> proposed triangle algebras and proved that there is a connection between these algebras and special residuated lattices. In 2010, Van Gasse et al.<sup>[9]</sup> explored various distinct filters of triangle algebras. In 2017, Zahiri et al.<sup>[10]</sup> characterized the structure of filters in triangle algebras. In 2021, Zahiri et al.<sup>[11]</sup> examined several crucial properties of distinct triangle algebras. In 2024, Zahiri et al.<sup>[12]</sup> introduced Stonean algebras and studied the association between the substructure of special triangle algebras. It can be observed that both triangle algebras and filters have a great significance in the study of fuzzy logics and logical algebras. Filters possess characteristics of non-classical algebraic logic and play a significant role in

Received 2024-08-08.

Sponsored by the Foreign Expert Program of China (Grant No. DL2023041002L) and the Yulin City Industry University Research Project (Grant No. CXY-2022-59).

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studying the completeness of logical systems<sup>[13-15]</sup>. From a logical perspective, all types of filters can be naturally regarded as a set of provable formulas. In fact, the notion of ideals in  $L$ -algebras is the same as the concept of filters in normal algebras. Therefore, by studying the structure and properties of triangle algebras and filters, we can extend these results to  $L$ -algebras, which is the motivations for our research. Under special conditions, triangle algebras are also triangle bounded  $L$ -algebras. That is, triangle bounded  $L$ -algebras are a generalization of triangle algebras, we can generalize some results of triangle algebras to triangle bounded  $L$ -algebras. Moreover, we can characterize the structure of triangle bounded  $L$ -algebras by defining triangle ideals and further study the corresponding logical systems through triangle bounded  $L$ -algebra and triangle ideals. By defining Stonean algebras, under special conditions, we can observe that the complement space of a triangle bounded  $L$ -algebras forms a complementary lattice.

The primary objective of this study is aimed at exploring triangle bounded  $L$ -algebras and triangle ideals. Firstly, we extend the definition of bounded  $L$ -algebra to the notion of triangle bounded  $L$ -algebras by adding a constant  $u \notin \{0, 1\}$  and two unary operations  $\nu$  and  $\mu$ . We present several examples of triangle bounded  $L$ -algebras with different conditions and explore their properties. Moreover, we investigate the connection between triangle bounded  $L$ -algebras with special conditions and  $u$ ,  $\nu$ , and  $\mu$ , respectively. Secondly, we give the concept of triangle ideals of triangle bounded  $L$ -algebras to explore the connection between the triangle ideal  $I$  of triangle bounded  $L$ -algebra  $L$  and the ideal  $I \cap E(L)$  of bounded  $L$ -algebra  $E(L)$ . In addition, by introducing the definition of Stonean triangle bounded  $L$ -algebras, we study their properties and the connection between Stonean triangle bounded  $L$ -algebras and Stonean triangle ideals. We classify and study various classes of triangle ideals, including Stonean triangle ideals, extended Stonean triangle ideals, and lattice ideals. Finally, we investigate the interrelationships among these various types of triangle ideals.

### 1 Preliminaries

**Definition 1.1**<sup>[1]</sup> For any  $\omega, \varphi, \psi \in L$ , if the algebra  $(L, \rightarrow, 1)$  satisfies the following conditions, then it is called an  $L$ -algebra.

- $(L_1) 1 \rightarrow \omega = \omega, \omega \rightarrow \omega = \omega \rightarrow 1 = 1;$
- $(L_2) (\omega \rightarrow \varphi) \rightarrow (\omega \rightarrow \psi) = (\varphi \rightarrow \omega) \rightarrow (\psi \rightarrow \omega);$
- $(L_3) \omega \rightarrow \varphi = \varphi \rightarrow \omega = 1$  implies  $\omega = \varphi.$

If the operation  $\rightarrow$  is taken as logical implicative, then there is a partial order on  $L$  defined by  $\varphi \leq \psi$  iff  $\varphi \rightarrow \psi = 1$ .

**Definition 1.2**<sup>[2]</sup> An  $L$ -algebra is bounded if it contains the smallest element 0. In a bounded  $L$ -algebra  $L$ , if the operation  $\prime: \omega \rightarrow \omega'$  is bijective, in which  $\omega' = \omega \rightarrow 0$  and  $\omega \in L$ , then we call  $L$  has negation  $\prime$ . The inverse mapping is denoted by  $\omega \rightarrow \omega \sim$ .

In Ref. [5], the  $L$ -algebra  $L$  has double negation, if  $\omega' = \omega \sim$  for any  $\omega \in L$ .

**Definition 1.3**<sup>[4]</sup> If the  $L$ -algebra  $L$  satisfies the condition  $(K): \omega \rightarrow (\varphi \rightarrow \omega) = 1$ , for any  $\omega, \varphi \in L$ , then it is called a  $KL$ -algebra.

**Definition 1.4**<sup>[4]</sup> If the  $L$ -algebra  $L$  satisfies the condition  $(C): (\varphi \rightarrow (\psi \rightarrow \omega)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \omega)) = 1$ , for any  $\psi, \omega, \varphi \in L$ , then it is called a  $CL$ -algebra.

In Ref. [4], we can find the  $CL$ -algebra is satisfied with condition  $(K)$ .

**Example 1.1**<sup>[5]</sup>  $(L = \{0, \varphi, \psi, 1\}, \rightarrow, 0, 1)$  is a bounded  $CL$ -algebra and also a  $KL$ -algebra, where the operation  $\rightarrow$  defined on  $L$  is presented in Table 1.

**Table 1** Cayley table for the binary operation  $\rightarrow$  on  $L$

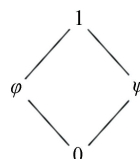
$\rightarrow$	0	$\varphi$	$\psi$	1
0	1	1	1	1
$\varphi$	0	1	1	1
$\psi$	0	$\varphi$	1	1
1	0	$\varphi$	$\psi$	1

**Definition 1.5**<sup>[7]</sup> In an  $L$ -algebra  $L$ , if  $((\varphi \rightarrow \psi) \rightarrow \omega) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \omega) = ((\varphi \rightarrow \psi) \rightarrow \omega) \rightarrow \omega$

for every  $\varphi, \psi, \omega \in L$ , then it is called a semiregular  $L$ -algebra.

**Proposition 1.2**<sup>[1]</sup> In an  $L$ -algebra  $L$ , if  $\varphi \leq \psi$ , then  $\omega \rightarrow \varphi \leq \omega \rightarrow \psi$  for any  $\varphi, \psi, \omega \in L$ .

**Example 1.2**<sup>[6]</sup>  $(L = \{0, \varphi, \psi, 1\}, \rightarrow, 0, 1)$  is a semiregular  $L$ -algebra, as shown in Fig. 1, in which the operation  $\rightarrow$  defined on  $L$  is presented in Table 2.



**Fig.1** Hasse diagram of  $L$

**Table 2** Cayley table for the binary operation  $\rightarrow$  on  $L$

$\rightarrow$	0	$\varphi$	$\psi$	1
0	1	1	1	1
$\varphi$	$\psi$	1	$\psi$	1
$\psi$	$\varphi$	$\varphi$	1	1
1	0	$\varphi$	$\psi$	1

**Proposition 1.2**<sup>[8]</sup> For any  $\varphi, \psi, \omega \in L$ , if  $L$  is an  $L$ -algebra, then the following conclusions are equivalent.

- 1)  $\varphi \leq \psi \rightarrow \varphi$  ;
- 2)  $\varphi \leq \psi \Rightarrow \psi \rightarrow \omega \leq \varphi \rightarrow \omega$  ;
- 3)  $((\varphi \rightarrow \psi) \rightarrow \omega) \rightarrow \omega \leq ((\varphi \rightarrow \psi) \rightarrow \omega) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \omega)$ .

We can find that a semiregular  $L$ -algebra is satisfied with condition (K).

**Definition 1.6**<sup>[1]</sup> For any  $\varphi, \psi \in L$ , if the subset  $I$  of  $L$ -algebra  $L$  is called an ideal, the following conditions are satisfied:

- 1)  $1 \in I$  ;
- 2)  $\varphi, \varphi \rightarrow \psi \in I \Rightarrow \psi \in I$  ;
- 3)  $\varphi \in I \Rightarrow (\varphi \rightarrow \psi) \rightarrow \psi \in I$  ;
- 4)  $\varphi \in I \Rightarrow \psi \rightarrow \varphi, \psi \rightarrow (\varphi \rightarrow \psi) \in I$ .

In Ref.[5], we can find that  $\{1\}$  and  $L$  are two ideals of  $L$ -algebra  $L$ . Moreover, if  $L$  is a  $KL$ -algebra, then 4) can be omitted, and if  $L$  is a  $CL$ -algebra, then 3), 4) can be omitted.

**Definition 1.7**<sup>[10]</sup> We call  $A = (A, \wedge, \vee, *, \Rightarrow, 0, 1)$  is a residuated lattice, if

- 1)  $(A, \wedge, \vee, 0, 1)$  is a bounded lattice;
- 2)  $(A, *, 1)$  is commutative monoid;
- 3)  $\varphi * \psi \leq \omega$  iff  $\varphi \leq \psi \Rightarrow \omega$  for any  $\varphi, \psi, \omega \in A$ .

**Definition 1.8**<sup>[16]</sup> In a residuated lattice  $A$ , if  $\varphi * (\varphi \Rightarrow \psi) = \varphi \wedge \psi$ , for any  $\varphi, \psi \in A$ , then  $A$  is called a divisible residuated lattice.

**Proposition 1.3**<sup>[16]</sup> A divisible residuated lattice is an  $L$ -algebra.

**Definition 1.9**<sup>[10]</sup> The structure  $A = (A, \wedge, \vee, *, \Rightarrow, \nu, \mu, 0, u, 1)$  is called a triangle algebra, if  $A$  can forms a residuated lattice, and for all  $\varphi, \psi \in A$ , we have:

- (T.1)  $\nu\varphi \leq \varphi$ , (T.1')  $\varphi \leq \mu\varphi$
- (T.2)  $\nu\varphi \leq \nu\nu\varphi$ , (T.2')  $\mu\mu\varphi \leq \mu\varphi$
- (T.3)  $\nu(\varphi \wedge \psi) = \nu\varphi \wedge \nu\psi$ , (T.3')  $\mu(\varphi \wedge \psi) = \mu\varphi \wedge \mu\psi$
- (T.4)  $\nu(\varphi \vee \psi) = \nu\varphi \vee \nu\psi$ , (T.4')  $\mu(\varphi \vee \psi) = \mu\varphi \vee \mu\psi$
- (T.5)  $\nu 1 = 1$ , (T.5')  $\mu 0 = 0$
- (T.6)  $\nu u = 0$ , (T.6')  $\mu u = 1$
- (T.7)  $\nu\mu\varphi = \mu\varphi$ , (T.7')  $\mu\nu\varphi = \nu\varphi$
- (T.8)  $\nu(\varphi \Rightarrow \psi) \leq \nu\varphi \Rightarrow \nu\psi$

$$(T.9) (\nu\varphi \Leftrightarrow \nu\psi) * (\mu\varphi \Leftrightarrow \mu\psi) \leq (\varphi \Leftrightarrow \psi),$$

where  $\varphi \Leftrightarrow \psi = (\varphi \Rightarrow \psi) * (\psi \Rightarrow \varphi)$

$$(T.10) \nu\varphi \Rightarrow \nu\psi \leq \nu(\nu\varphi \Rightarrow \nu\psi)$$

## 2 Triangle Bounded $L$ -algebras

By generalizing the notion of bounded  $L$ -algebra, that is, by introducing a constant  $u$  (distinct from 0 and 1) and two unary operations  $\nu$  and  $\mu$ , we obtain the definition of triangle bounded  $L$ -algebras. In addition, some examples of triangle bounded  $L$ -algebras with different conditions are illustrated, and the properties of these algebras are also studied.

**Definition 2.1** The structure  $(L, \rightarrow, \nu, \mu, 0, u, 1)$  is called a triangle bounded  $L$ -algebra, if  $(L, \rightarrow, 0, 1)$  is a bounded  $L$ -algebra, and for any  $\varphi, \omega \in L$ , we have:

- (TL1)  $\nu\varphi \leq \varphi$ , (TL1')  $\varphi \leq \mu\varphi$
- (TL2)  $\nu\varphi \leq \nu\nu\varphi$ , (TL2')  $\mu\mu\varphi \leq \mu\varphi$
- (TL3)  $\nu 1 = 1$ , (TL3')  $\mu 0 = 0$
- (TL4)  $\nu u = 0$ , (TL4')  $\mu u = 1$
- (TL5)  $\nu\mu\varphi = \mu\varphi$ , (TL5')  $\mu\nu\varphi = \nu\varphi$
- (TL6) If  $\varphi \leq \omega$ , then  $\mu\varphi \leq \mu\omega$
- (TL7)  $\nu(\varphi \rightarrow \omega) \leq \nu\varphi \rightarrow \nu\omega$
- (TL8) If  $\nu\varphi = \nu\omega$ ,  $\mu\varphi = \mu\omega$ , then  $\varphi = \omega$
- (TL9)  $\nu\varphi \rightarrow \nu\omega \leq \nu(\nu\varphi \rightarrow \nu\omega)$

From (TL1) and (TL2), it is obvious that  $\nu\nu\varphi = \nu\varphi$  in a triangle bounded  $L$ -algebra  $L$ . Similarly, from (TL1') and (TL2'), we have  $\mu\mu\varphi = \mu\varphi$ . From (TL1) and (TL9),  $\nu\varphi \rightarrow \nu\omega = \nu(\nu\varphi \rightarrow \nu\omega)$  can be gotten. (TL6) and (TL7) implies that  $\nu$  and  $\mu$  are increasing operations. According to the conditions (TL1'), (TL2'), and (TL6), it can be inferred that  $\varphi \leq \mu\mu\varphi = \mu\varphi$  and if  $\varphi \leq \omega$ , then  $\mu\varphi \leq \mu\omega$ , for any  $\varphi, \omega \in L$ . Therefore,  $\mu$  is a closure on  $L$ . According to (TL1), (TL2), and (TL7), it can be inferred that  $\nu\nu\varphi = \nu\varphi \leq \varphi$  and if  $\varphi \leq \omega$ , then  $\nu\varphi \leq \nu\omega$ , for any  $\varphi, \omega \in L$ . Therefore,  $\nu$  is a dual closure on  $L$ .

If the triangle bounded  $L$ -algebra  $L$  satisfies conditions (K), (C), or it is a semiregular  $L$ -algebra, then it is called a triangle bounded  $KL$ -algebra, triangle bounded  $CL$ -algebra or a triangle bounded semiregular  $L$ -algebra, respectively.

**Example 2.1**<sup>[10]</sup> Let  $L = ([0, 1], \wedge_L, \vee_L, *, \rightarrow, 0, 1)$  be a divisible residuated lattice, where  $\varphi \vee_L \omega = \max\{\varphi, \omega\}$ ,  $\varphi \wedge_L \omega = \min\{\varphi, \omega\}$ ,  $\varphi * \omega = \min\{\varphi, \omega\}$ , if  $\varphi \leq \omega$ , then  $\varphi \rightarrow \omega = 1$ ; if  $\omega \leq \varphi$ , then  $\varphi \rightarrow \omega = \omega$ , for any  $\varphi, \omega \in L$ .

Then for

$$\text{Int}(L) = \{ [\varphi_1, \varphi_2] : (\varphi_1, \varphi_2) \in L \times L, \varphi_1 \leq \varphi_2 \}$$

We define:

$$[\varphi_1, \varphi_2] \odot [\omega_1, \omega_2] = [\varphi_1 * \omega_1, \varphi_2 * \omega_2]$$

$$[\varphi_1, \varphi_2] \Rightarrow [\omega_1, \omega_2] =$$

$$[(\varphi_1 \rightarrow \omega_1) \wedge (\varphi_2 \rightarrow \omega_2), \varphi_2 \rightarrow \omega_2]$$

$$[\varphi_1, \varphi_2] \wedge [\omega_1, \omega_2] = [\varphi_1 \wedge_L \omega_1, \varphi_2 \wedge_L \omega_2]$$

$$[\varphi_1, \varphi_2] \vee [\omega_1, \omega_2] = [\varphi_1 \vee_L \omega_1, \varphi_2 \vee_L \omega_2]$$

For any  $[\varphi_1, \varphi_2], [\omega_1, \omega_2] \in \text{Int}(L)$ . The structure  $\text{Int}(L) = (\text{Int}(L), \wedge, \vee, \odot, \Rightarrow, [0,0], [1,1])$  is also a residuated lattice.

We also define a partial order relation in  $\text{Int}(L)$ ,  $[\varphi_1, \varphi_2] \leq [\omega_1, \omega_2] \Leftrightarrow \varphi_1 \leq \omega_1$  and  $\varphi_2 \leq \omega_2$ , for any  $[\varphi_1, \varphi_2], [\omega_1, \omega_2] \in \text{Int}(L)$ . If we define  $\nu[\varphi_1, \varphi_2] = [\varphi_1, \varphi_1]$ , and  $\mu[\varphi_1, \varphi_2] = [\varphi_2, \varphi_2]$ ,  $u = [0,1]$ , for any  $[\varphi_1, \varphi_2] \in \text{Int}(L)$ , then

$$\text{Int}(L) = (\text{Int}(L), \wedge, \vee, \odot, \Rightarrow,$$

$$\nu, \mu, [0,0], u, [1,1])$$

is a triangle algebra.

Moreover, we can conclude that  $\text{Int}(L)$  is also a divisible residuated lattice. This is because, for any  $[\varphi_1, \varphi_2], [\omega_1, \omega_2] \in \text{Int}(L)$ , if there is an order relation between  $[\varphi_1, \varphi_2]$  and  $[\omega_1, \omega_2]$ , we can get that  $[\varphi_1, \varphi_2] \leq [\omega_1, \omega_2]$  or  $[\varphi_1, \varphi_2] \geq [\omega_1, \omega_2]$ . If  $[\varphi_1, \varphi_2] \leq [\omega_1, \omega_2]$ , then

$$\begin{aligned} [\varphi_1, \varphi_2] \wedge [\omega_1, \omega_2] &= \\ [\varphi_1 \wedge_L \omega_1, \varphi_2 \wedge_L \omega_2] &= \\ [\varphi_1, \varphi_2] &= [\varphi_1 * 1, \varphi_2 * 1] = [\varphi_1, \varphi_2] \odot \\ [1, 1] &= [\varphi_1, \varphi_2] \odot [(\varphi_1 \rightarrow \omega_1) \wedge_L \\ (\varphi_2 \rightarrow \omega_2), \varphi_2 \rightarrow \omega_2] &= [\varphi_1, \varphi_2] \odot \\ ([\varphi_1, \varphi_2] \Rightarrow [\omega_1, \omega_2]) & \end{aligned}$$

We can use the same method to prove that if  $[\varphi_1, \varphi_2] \geq [\omega_1, \omega_2]$ , then  $[\varphi_1, \varphi_2] \wedge [\omega_1, \omega_2] = [\varphi_1, \varphi_2] \odot ([\varphi_1, \varphi_2] \Rightarrow [\omega_1, \omega_2])$ .

If there is no order relation between  $[\varphi_1, \varphi_2]$  and  $[\omega_1, \omega_2]$ , then we can get that  $\varphi_1 < \omega_1$  and  $\varphi_2 > \omega_2$ , or  $\varphi_1 > \omega_1$  and  $\varphi_2 < \omega_2$ . If  $\varphi_1 < \omega_1$  and  $\varphi_2 > \omega_2$ , then

$$\begin{aligned} [\varphi_1, \varphi_2] \wedge [\omega_1, \omega_2] &= \\ [\varphi_1 \wedge_L \omega_1, \varphi_2 \wedge_L \omega_2] &= \\ [\varphi_1, \omega_2] &= [\varphi_1 * \omega_2, \varphi_2 * \omega_2] = \\ [\varphi_1, \varphi_2][1 \wedge_L \omega_2, \omega_2] &= \\ [\varphi_1, \varphi_2] \odot [(\varphi_1 \rightarrow \omega_1) \wedge_L (\varphi_2 \rightarrow \omega_2), & \\ \varphi_2 \rightarrow \omega_2] &= [\varphi_1, \varphi_2] \odot \\ ([\varphi_1, \varphi_2] \Rightarrow [\omega_1, \omega_2]) & \end{aligned}$$

We can use the same method to prove that when  $\varphi_1 > \omega_1$  and  $\varphi_2 < \omega_2$ , we have

$$\begin{aligned} [\varphi_1, \varphi_2] \wedge [\omega_1, \omega_2] &= [\varphi_1, \varphi_2] \odot \\ ([\varphi_1, \varphi_2] \Rightarrow [\omega_1, \omega_2]) & \end{aligned}$$

Therefore,  $\text{Int}(L)$  is a divisible residuated lattice

(as shown in Fig.2), according to Definition 1.8.

It is easily evident that  $\text{Int}(L) = (\text{Int}(L), \wedge, \vee, \odot, \Rightarrow, \nu, \mu, [0,0], u, [1,1])$  is also a triangle bounded  $L$ -algebras, according to Definition 2.1 and Proposition 1.3.

Consequently, triangle bounded  $L$ -algebras are generalization of triangle algebras.

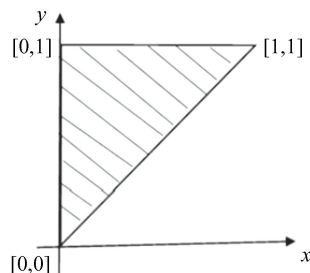


Fig.2 Hasse diagram of  $\text{Int}(L)$

**Remark 2.1** If  $L = (L, \rightarrow, \nu, \mu, 0, u, 1)$  is a triangle bounded  $L$ -algebra, then  $\nu\varphi = \nu\omega$  and  $\mu\varphi = \mu\omega$  iff  $\varphi = \omega$  for any  $\varphi, \omega \in L$ .

**Proof** For any  $\varphi, \omega \in L$ , according to (TL8), if  $\nu\varphi = \nu\omega$  and  $\mu\varphi = \mu\omega$ , then  $\varphi = \omega$ . If  $\varphi = \omega$ , according to (TL6), then  $\mu\varphi \leq \mu\omega, \mu\omega \leq \mu\varphi$ . Therefore, according to (L3),  $\mu\varphi = \mu\omega$ . And according to (TL3) and (TL7), there is  $1 = \nu 1 = \nu(\varphi \rightarrow \omega) \leq \nu\varphi \rightarrow \nu\omega$  and  $1 = \nu 1 = \nu(\omega \rightarrow \varphi) \leq \nu\omega \rightarrow \nu\varphi$ . Therefore, according to (L3),  $\nu\varphi = \nu\omega$ . Consequently, if  $\varphi = \omega$ , then  $\nu\varphi = \nu\omega$  and  $\mu\varphi = \mu\omega$ . Therefore,  $\nu\varphi = \nu\omega$  and  $\mu\varphi = \mu\omega$  iff  $\varphi = \omega$  for any  $\varphi, \omega \in L$ .

In a special case, we can get that a triangle bounded  $L$ -algebra solely through the operation  $\nu$ .

**Proposition 2.1** Let  $(L, \rightarrow, 0, 1)$  be a bounded  $KL$ -algebra with double negation,  $u' = u$  for exists  $u \in L$ , and  $\nu$  satisfies (TL1) – (TL5) and (TL7) – (TL9). If we define  $\mu\varphi = (\nu\varphi)'$  for any  $\varphi \in L$ , then  $(L, \rightarrow, \nu, \mu, 0, u, 1)$  is a triangle bounded  $L$ -algebra.

**Proof** Let  $(L, \rightarrow, 0, 1)$  be a bounded  $KL$ -algebra with double negation. According to (TL1), we can get that  $\nu\varphi' \leq \varphi'$ , and so  $\varphi = \varphi'' \leq (\nu\varphi')' = \mu\varphi$ . So (TL1') holds.

We can get that  $\nu\varphi' = \nu(\nu\varphi') = \nu(\nu\varphi')''$ , according to (TL1), (TL2) and double negation. Therefore  $\mu\mu\varphi = (\nu(\nu\varphi')'' )' = (\nu(\nu\varphi'))' = (\nu\varphi')' = \mu\varphi$  So (TL2') holds.

$$\mu 0 = (\nu 0')' = (\nu 1)' = 1' = 0$$

So (TL3') holds.

Using the condition  $u' = u$ , we can conclude that

$\mu u = (\nu u')' = (\nu u)' = 0' = 1$ , by (TL4). So (TL4') holds.

Because  $\nu\mu\varphi' = \mu\varphi'$  by (TL5), we can get that  $\nu(\nu\varphi)' = \nu(\nu\varphi'')' = (\nu\varphi)'$ , and so  $\mu\nu\varphi = (\nu(\nu\varphi)')' = (\nu\varphi)'' = \nu\varphi$ . So (TL5') holds.

If  $\varphi \leq \omega$ , then  $1 = \nu 1 = \nu(\omega' \rightarrow \varphi') \leq \nu\omega' \rightarrow \nu\varphi'$ , since  $\omega' \leq \varphi'$ . So we have  $\nu\omega' \leq \nu\varphi'$ . We can conclude that  $\mu\varphi = (\nu\varphi')' \leq (\nu\omega')' = \mu\omega$ . So (TL6) holds.

Therefore,  $(L, \rightarrow, \nu, \mu, 0, u, 1)$  is a triangle bounded  $L$ -algebra.

**Definition 2.2** In a triangle bounded  $L$ -algebra  $L = (L, \rightarrow, \nu, \mu, 0, u, 1)$ , we set  $E(L) = \{\varphi \in L \mid \nu\varphi = \varphi\}$ ,  $\nu(L) = \{\nu\varphi \mid \varphi \in L\}$ , and  $\mu(L) = \{\mu\varphi \mid \varphi \in L\}$ .

It can be observed that  $\nu\varphi, \mu\varphi \in E(L)$ , since  $\nu\nu\varphi = \varphi$  and  $\nu\mu\varphi = \mu\varphi$ , for all  $\varphi \in L$ .

**Proposition 2.2** If  $L = (L, \rightarrow, \nu, \mu, 0, u, 1)$  is a triangle bounded  $L$ -algebra, then

$$E(L) = \nu(L) = \mu(L) = \nu(E(L)) = \mu(E(L))$$

**Proof** If  $\varphi \in E(L) \subseteq L$ , then  $\varphi = \nu\varphi \in \nu(L)$  for all  $\varphi \in L$ . Therefore,  $E(L) \subseteq \nu(L)$ .

If  $\varphi \in \nu(L) \subseteq L$ , then  $\varphi = \nu\omega = \mu\nu\omega \in \mu(L)$ , since (TL5') and  $\nu\omega \in E(L)$ , for some  $\omega \in L$ . Therefore,  $\nu(L) \subseteq \mu(L)$ .

If  $\varphi \in \mu(L)$ , then  $\varphi = \mu\omega = \nu\mu\omega \in \nu(E(L))$ , since (TL5) and  $\mu\omega \in E(L)$ , for some  $\omega \in L$ . Therefore,  $\mu(L) \subseteq \nu(E(L))$ .

If  $\varphi \in \nu(E(L))$ , then  $\varphi = \nu\omega = \mu\nu\omega \in \mu(E(L))$  for some  $\omega \in E(L)$  by (TL5'). Therefore,  $\nu(E(L)) \subseteq \mu(E(L))$ .

If  $\varphi \in \mu(E(L))$ , then  $\varphi = \mu\omega \in \mu(E(L))$  for some  $\omega \in E(L)$ . We can get that  $\varphi = \mu\omega \in E(L)$ , since  $\nu\mu\omega = \mu\omega$ .

Therefore,  $\mu(E(L)) \subseteq E(L). E(L) = \nu(E(L)) = \nu(L) = \mu(L) = \mu(E(L))$ .

**Corollary 2.1** If  $L$  is a triangle bounded  $L$ -algebra, then  $E(L) = \{\varphi \in L \mid \mu\varphi = \varphi\}$  and  $(E(L), \rightarrow, 0, 1)$  constitutes a bounded  $L$ -algebra.

**Proof** Let  $L$  be a triangle bounded  $L$ -algebra. For any  $\varphi, \psi \in E(L) \subseteq L$ , we have  $\mu\varphi = \mu\nu\varphi = \nu\varphi = \varphi$ , by Definition 2.1 and 2.2. Therefore,  $\{\varphi \in L \mid \mu\varphi = \varphi\}$ . According to (TL1) and (TL9), we can get that  $\nu\varphi \rightarrow \nu\psi = \varphi \rightarrow \psi$  and so  $\nu(\nu\varphi \rightarrow \nu\psi) = \nu(\varphi \rightarrow \psi)$ . Therefore,  $\varphi \rightarrow \psi = \nu(\varphi \rightarrow \psi)$  and so  $\varphi \rightarrow \psi \in E(L)$ . Moreover,  $0, 1 \in E(L)$ , since  $\nu 0 = 0$  and  $\nu 1 = 1$ . Consequently,  $(E(L), \rightarrow, 0, 1)$  constitutes a bounded  $L$ -algebra.

**Proposition 2.3** If  $L$  is a triangle bounded  $L$ -algebra, then  $\nu\varphi = \sup\{\omega \in E(L) \mid \omega \leq \varphi\}$  and  $\mu\varphi = \inf\{\omega \in E(L) \mid \varphi \leq \omega\}$ , for any  $\varphi \in L$ .

**Proof** If  $L$  is a triangle bounded  $L$ -algebra. Because  $\nu\nu\varphi = \nu\varphi \leq \varphi$  for any  $\varphi \in L$ , so  $\nu\varphi \in \{\omega \in E(L) \mid \omega \leq \varphi\}$ , we have  $\nu\varphi \leq \sup\{\omega \in E(L) \mid \omega \leq \varphi\}$ . Moreover,  $\nu a \leq \nu\varphi$ , since  $\nu$  is increasing operation, for every  $a \in \{\omega \in E(L) \mid \omega \leq \varphi\}$ . Hence,  $\sup\{\omega \in E(L) \mid \omega \leq \varphi\} \leq \nu\varphi$ . Consequently,  $\nu\varphi = \sup\{\omega \in E(L) \mid \omega \leq \varphi\}$ .

We can get that  $\mu\mu\varphi = \mu\varphi \geq \varphi$  for any  $\varphi \in L$ . Therefore, we have  $\mu\varphi \in \{\omega \in E(L) \mid \varphi \leq \omega\}$  and  $\mu\varphi \geq \inf\{\omega \in E(L) \mid \varphi \leq \omega\}$ . Moreover,  $\inf\{\omega \in E(L) \mid \varphi \leq \omega\} \geq \mu\varphi$ , since  $\varphi \leq a \Rightarrow \mu\varphi \leq \mu a = a$  for every  $a \in \{\omega \in E(L) \mid \varphi \leq \omega\}$ , by Corollary 2.1. Therefore,  $\mu\varphi = \inf\{\omega \in E(L) \mid \varphi \leq \omega\}$ .

Given a bounded  $L$ -algebra, it is possible to generate some triangle bounded  $L$ -algebras. Some examples will be given to illustrate it.

**Example 2.2** If  $L = (\{0, a, 1\}, \rightarrow_L, 0, 1)$  is a bounded  $L$ -algebra, then  $A = (A, \rightarrow, \nu, \mu, [0, 0], [0, 1], [1, 1])$  is a triangle bounded  $L$ -algebra, in which,  $A = \{[0, 0], [0, 1], [a, a], [1, 1]\}$ ,  $\nu[\varrho_1, \varrho_2] = [\varrho_1, \varrho_1]$ , and  $\mu[\varrho_1, \varrho_2] = [\varrho_2, \varrho_2]$ , for any  $[\varrho_1, \varrho_2], [\varphi_1, \varphi_2] \in A$ . The implication operations defined on  $L$  and  $A$  are presented in Tables 3 and 4, respectively.

$$[\varrho_1, \varrho_2] \rightarrow [\varphi_1, \varphi_2] = \begin{cases} [\varrho_1 \rightarrow_L \varphi_1, \varrho_2 \rightarrow_L \varphi_2], \\ \text{if } (\varrho_1 \rightarrow_L \varphi_1) \leq (\varrho_2 \rightarrow_L \varphi_2) \\ [\varrho_2 \rightarrow_L \varphi_2, \varrho_2 \rightarrow_L \varphi_2], \text{ else} \end{cases}$$

Table 3 Cayley table for the implication operation of  $L$

$\rightarrow_L$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

Table 4 Cayley table for the implication operation of  $A$

$\rightarrow$	$[0, 0]$	$[a, a]$	$[0, 1]$	$[1, 1]$
$[0, 0]$	$[1, 1]$	$[1, 1]$	$[1, 1]$	$[1, 1]$
$[a, a]$	$[0, 0]$	$[1, 1]$	$[0, 1]$	$[1, 1]$
$[0, 1]$	$[0, 0]$	$[a, a]$	$[1, 1]$	$[1, 1]$
$[1, 1]$	$[0, 0]$	$[a, a]$	$[0, 1]$	$[1, 1]$

It is obviously that  $A = (A, \rightarrow, [0, 0], [1, 1])$  is a bounded  $L$ -algebra. Moreover, for any  $[\varrho_1, \varrho_2], [\varphi_1, \varphi_2] \in A$ , we can find that  $\varrho_1 \leq \varrho_2$  in  $[\varrho_1, \varrho_2]$  and  $[\varrho_1, \varrho_2] \leq [\varphi_1, \varphi_2]$  iff  $\varrho_1 \leq \varphi_1$  and  $\varrho_2 \leq \varphi_2$ .

We can conclude that  $\nu[\varrho_1, \varrho_2] = [\varrho_1, \varrho_1] \leq [\varrho_1, \varrho_2]$ ,  $\mu[\varrho_1, \varrho_2] = [\varrho_2, \varrho_2] \geq [\varrho_1, \varrho_2]$ ,  $\nu[\varrho_1, \varrho_2] = [\varrho_1, \varrho_2] = \nu\nu[\varrho_1, \varrho_2]$ ,  $\mu[\varrho_1, \varrho_2] = [\varrho_1, \varrho_2] = \mu\mu[\varrho_1, \varrho_2]$ ,

$\nu[1,1] = [1,1]$ ,  $\mu[0,0] = [0,0]$ ,  $\nu\mu[\varrho_1, \varrho_2] = \nu[\varrho_2, \varrho_2] = \mu[\varrho_1, \varrho_2]$ ,  $\nu[0,1] = [0,0]$ ,  $\mu[0,1] = [1,1]$ , and  $\mu\nu[\varrho_1, \varrho_2] = \mu[\varrho_1, \varrho_1] = [\varrho_1, \varrho_1] = \nu[\varrho_1, \varrho_2]$ . So (TL1)–(TL5) and (TL1')–(TL5') holds.

We can observe that in  $A$ , if  $[\varrho_1, \varrho_2] \leq [\varphi_1, \varphi_2]$ , then  $\mu[\varrho_1, \varrho_2] = [\varrho_2, \varrho_2] \leq [\varphi_1, \varphi_2] = \mu[\varphi_1, \varphi_2]$ . So (TL6) holds.

For (TL7). If  $\varrho_1 \rightarrow \varphi_1 \leq \varrho_2 \rightarrow \varphi_2$ , then

$$\begin{aligned} \nu([\varrho_1, \varrho_2] \rightarrow [\varphi_1, \varphi_2]) &= \nu([\varrho_1, \varrho_2] \rightarrow [\varphi_1, \varphi_2]) = \\ &= \nu[\varrho_1 \rightarrow_L \varphi_1, \varrho_2 \rightarrow_L \varphi_2] = [\varrho_1 \rightarrow_L \varphi_1, \varrho_1 \rightarrow_L \varphi_1] = \\ &= [\varrho_1, \varrho_1] \rightarrow [\varphi_1, \varphi_1] = \nu[\varrho_1, \varrho_1] \rightarrow \nu[\varphi_1, \varphi_2] \end{aligned}$$

If  $\varrho_1 \rightarrow \varphi_1 \not\leq \varrho_2 \rightarrow \varphi_2$ , in fact, this situation will result in  $0 \rightarrow 0 \not\leq 1 \rightarrow 0$ , and  $0 \rightarrow a \not\leq 1 \rightarrow a$ , because  $0 \rightarrow 0 = 1 \not\leq 0 = 1 \rightarrow 0$  and  $0 \rightarrow a = 1 \not\leq a = 1 \rightarrow a$ .

We can find that

$$\begin{aligned} \nu([0,1] \rightarrow [0,0]) &= \nu[0,0] = [0,0] \leq [1,1] = \\ &= [0,0] \rightarrow [0,0] = \nu[0,1] \rightarrow \nu[0,0] \end{aligned}$$

and

$$\begin{aligned} \nu([0,1] \rightarrow [a,a]) &= \nu[a,a] = [a,a] \leq [1,1] = \\ &= [0,0] \rightarrow [a,a] = \nu[0,1] \rightarrow \nu[a,a] \end{aligned}$$

Therefore,

$$\begin{aligned} \nu([\varrho_1, \varrho_2] \rightarrow \nu[\varphi_1, \varphi_2]) &\leq \\ \nu[\varrho_1, \varrho_2] \rightarrow \nu[\varphi_1, \varphi_2] \end{aligned}$$

for any  $[\varrho_1, \varrho_2], [\varphi_1, \varphi_2] \in A$ . So (TL7) holds.

If  $\nu[\varrho_1, \varrho_2] = \nu[\varphi_1, \varphi_2]$ ,  $\mu[\varrho_1, \varrho_2] = \mu[\varphi_1, \varphi_2]$ , then  $\nu[\varrho_1, \varrho_2] = [\varrho_1, \varrho_1] = [\varphi_1, \varphi_1] = \nu[\varphi_1, \varphi_2]$  and  $\mu[\varrho_1, \varrho_2] = [\varrho_2, \varrho_2] = [\varphi_2, \varphi_2] = \mu[\varphi_1, \varphi_2]$ . Therefore,  $[\varrho_1, \varrho_2] = [\varphi_1, \varphi_2]$ . So (TL8) holds.

$$\begin{aligned} \nu[\varrho_1, \varrho_2] \rightarrow \nu[\varphi_1, \varphi_2] &= [\varrho_1, \varrho_1] \rightarrow [\varphi_1, \varphi_1] = \\ &= [\varrho_1 \rightarrow \varphi_1, \varrho_1 \rightarrow \varphi_1] = \nu[\varrho_1 \rightarrow \varphi_1, \varrho_1 \rightarrow \varphi_1] = \\ &= \nu([\varrho_1, \varrho_1] \rightarrow [\varphi_1, \varphi_1]) = \nu(\nu[\varrho_1, \varrho_2] \rightarrow \\ &= \nu[\varphi_1, \varphi_2]) \end{aligned}$$

So (TL9) holds.

Therefore, we can conclude that  $A$  is a triangle bounded  $L$ -algebra.

**Example 2.3** If  $L = (\{0, \varphi, \psi, \omega, 1\}, \rightarrow_L, 0, 1)$  is a bounded  $L$ -algebra, then  $B = (B, \rightarrow, \nu, \mu, [0, 0], [0, 1], [1, 1])$  is a triangle bounded  $L$ -algebra, in which,  $B = \{[0, 0], [0, 1], [\varphi, \varphi], [\psi, \psi], [\omega, \omega], [1, 1]\}$ ,  $\nu[\psi_1, \psi_2] = [\psi_1, \psi_1]$ , and  $\mu[\psi_1, \psi_2] = [\psi_2, \psi_2]$ , for any  $[\psi_1, \psi_2], [\omega_1, \omega_2] \in B$ . The implication operations defined on  $L$  and  $B$  are presented in Tables 5 and 6, respectively.

$$\begin{aligned} [\psi_1, \psi_2] \rightarrow [\omega_1, \omega_2] &= \\ \left\{ \begin{aligned} &[\psi_1 \rightarrow_L \omega_1, \psi_2 \rightarrow_L \omega_2], \\ &\text{if } (\psi_1 \rightarrow_L \omega_1) \leq (\psi_2 \rightarrow_L \omega_2) \\ &[\psi_2 \rightarrow_L \omega_2, \psi_2 \rightarrow_L \omega_2], \text{ else} \end{aligned} \right. \end{aligned}$$

Table 5 Cayley table for the implication operation on  $L$

$\rightarrow_L$	0	$\varphi$	$\psi$	$\omega$	1
0	1	1	1	1	1
$\varphi$	0	$\varphi$	1	1	1
$\psi$	0	$\varphi$	1	1	1
$\omega$	0	$\varphi$	$\psi$	1	1
1	0	$\varphi$	$\psi$	$\omega$	1

Table 6 Cayley table for the implication operation of  $B$

$\rightarrow$	$[0,0]$	$[\varphi, \varphi]$	$[\psi, \psi]$	$[\omega, \omega]$	$[0,1]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[\varphi, \varphi]$	$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$	$[0,1]$	$[1,1]$
$[\psi, \psi]$	$[0,0]$	$[\varphi, \varphi]$	$[1,1]$	$[1,1]$	$[0,1]$	$[1,1]$
$[\omega, \omega]$	$[0,0]$	$[\varphi, \varphi]$	$[\psi, \psi]$	$[1,1]$	$[0,1]$	$[1,1]$
$[0,1]$	$[0,0]$	$[\varphi, \varphi]$	$[\psi, \psi]$	$[\omega, \omega]$	$[1,1]$	$[1,1]$
$[1,1]$	$[0,0]$	$[\varphi, \varphi]$	$[\psi, \psi]$	$[\omega, \omega]$	$[0,1]$	$[1,1]$

It is obviously that  $B = (B, \rightarrow, [0, 0], [1, 1])$  is a bounded  $L$ -algebra. We can prove that (TL1)–(TL6), (TL1')–(TL5'), and (TL8)–(TL9) hold, according to Example 2.2.

For (TL7). If  $\psi_1 \rightarrow \omega_1 \leq \psi_2 \rightarrow \omega_2$ , then

$$\begin{aligned} \nu([\psi_1, \psi_2] \rightarrow [\omega_1, \omega_2]) &= \nu[\psi_1 \rightarrow_L \omega_1, \psi_2 \rightarrow_L \omega_2] = \\ &= [\psi_1 \rightarrow_L \omega_1, \psi_1 \rightarrow_L \omega_1] = [\psi_1, \psi_1] \rightarrow [\omega_1, \omega_1] = \\ &= \nu[\psi_1, \psi_2] \rightarrow \nu[\omega_1, \omega_2] \end{aligned}$$

For any  $[\psi_1, \psi_2], [\omega_1, \omega_2] \in B$ . If  $\psi_1 \rightarrow \omega_1 \not\leq \psi_2 \rightarrow \omega_2$ , in fact, this situation will result in  $0 \rightarrow 0 \not\leq 1 \rightarrow 0$  and  $0 \rightarrow \sigma \not\leq 1 \rightarrow \sigma$ , where  $\sigma \in \{\varphi, \psi, \omega\}$ , because  $0 \rightarrow 0 = 1 \not\leq 0 = 1 \rightarrow 0$  and  $0 \rightarrow \sigma = 1 \not\leq \sigma = 1 \rightarrow \sigma$ . We can get that

$$\begin{aligned} \nu([0,1] \rightarrow [0,0]) &= \nu[0,0] = [0,0] \leq [1,1] = \\ &= [0,0] \rightarrow [0,0] = \nu[0,1] \rightarrow \nu[0,0] \end{aligned}$$

and

$$\begin{aligned} \nu([0,1] \rightarrow [\sigma, \sigma]) &= \nu[\sigma, \sigma] = [\sigma, \sigma] \leq [1,1] = \\ &= [0,0] \rightarrow [\sigma, \sigma] = \nu[0,1] \rightarrow \nu[\sigma, \sigma] \end{aligned}$$

Therefore,

$$\nu([\psi_1, \psi_2] \rightarrow [\omega_1, \omega_2]) \leq \nu[\psi_1, \psi_2] \rightarrow \nu[\omega_1, \omega_2]$$

For any  $[\psi_1, \psi_2], [\omega_1, \omega_2] \in B$ . So (TL7) holds.

Consequently,  $B$  is a triangle bounded  $L$ -algebra.

We can give an infinite example of triangle bounded  $L$ -algebras, according to Example 2.3. If  $L = (\{0, s, t, \dots, n, \dots, 1\}, \rightarrow_L, 0, 1)$  is a bounded  $L$ -algebra, then  $B = (B, \rightarrow, \nu, \mu, [0, 0], [0, 1], [1, 1])$  is a triangle bounded  $L$ -algebra, in which,

$$\begin{aligned} B &= \{[0, 0], [0, 1], [s, s], [t, t], \dots, \\ &= [n, n], \dots, [1, 1]\} \end{aligned}$$

$\nu[\varphi_1, \varphi_2] = [\varphi_1, \varphi_1]$ , and  $\mu[\varphi_1, \varphi_2] = [\varphi_2, \varphi_2]$ , for any  $[\varphi_1, \varphi_2] \in B$ .

**Example 2.4** If  $L = (\{0, a, 1\}, \rightarrow_L, 0, 1)$  is a bounded  $L$ -algebra, then  $C = (C, \rightarrow, \nu, \mu, [0, a], [0,$

$1], [1,1])$  is a triangle bounded  $L$ -algebra, where  $C = \{[0,a], [a,a], [0,1], [1,1]\}$ ,  $\nu[0,1] = \nu[0,a] = \mu[0,a] = \nu[a,a] = [0,a]$ ,  $\mu[a,a] = [a,a]$ , and  $\mu[1,1] = \mu[0,1] = \nu[1,1] = [1,1]$ . The implication operations defined on  $L$  and  $C$  are presented in Tables 7 and 8, respectively.

$$[\psi_1, \psi_2] \rightarrow [\omega_1, \omega_2] = \begin{cases} [\psi_1 \rightarrow_L \omega_1, \psi_2 \rightarrow_L \omega_2], \\ \text{if } (\psi_1 \rightarrow_L \omega_1) \leq (\psi_2 \rightarrow_L \omega_2) \\ [\psi_2 \rightarrow_L \omega_2, \psi_1 \rightarrow_L \omega_1], \text{ else} \end{cases}$$

**Table 7** Cayley table for the implication operation of  $L$

$\rightarrow_L$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

**Table 8** Cayley table for the implication operation of  $C$

$\rightarrow$	$[0,a]$	$[a,a]$	$[0,1]$	$[1,1]$
$[0,a]$	$[1,1]$	$[1,1]$	$[1,1]$	$[1,1]$
$[a,a]$	$[0,1]$	$[1,1]$	$[0,1]$	$[1,1]$
$[0,1]$	$[a,a]$	$[a,a]$	$[1,1]$	$[1,1]$
$[1,1]$	$[0,a]$	$[a,a]$	$[0,1]$	$[1,1]$

It is obviously that  $C = (C, \rightarrow, [0,a], [1,1])$  is a semiregular  $L$ -algebra with negation. Moreover, we can find that  $\psi_1 \leq \psi_2$  for any  $[\psi_1, \psi_2] \in C$ .

We use the same method as in Example 2.2, We can find that (TL1)–(TL6), (TL8), and (TL1')–(TL5') hold.

For any  $[\psi_1, \psi_2], [\omega_1, \omega_2] \in C$ . If  $\psi_1 \rightarrow \omega_1 \leq \psi_2 \rightarrow \omega_2$ , then  $\nu([\psi_1, \psi_2] \rightarrow [\omega_1, \omega_2]) = \nu[\psi_1 \rightarrow_L \omega_1, \psi_2 \rightarrow_L \omega_2] = [\psi_1 \rightarrow_L \omega_1, \psi_1 \rightarrow_L \omega_1] = [\psi_1, \psi_1] \rightarrow [\omega_1, \omega_1] = \nu[\psi_1, \psi_2] \rightarrow \nu[\omega_1, \omega_2]$ . If  $\psi_1 \rightarrow \omega_1 \not\leq \psi_2 \rightarrow \omega_2$ , in fact, this situation will result in  $0 \rightarrow 0 \not\leq 1 \rightarrow a$  and  $0 \rightarrow a \not\leq 1 \rightarrow a$ , because  $0 \rightarrow 0 = 1 \not\leq a = 1 \rightarrow a$  and  $0 \rightarrow a = 1 \not\leq a = 1 \rightarrow a$ . We can find that

$$\nu([0,1] \rightarrow [0,a]) = \nu[a,a] = [0,a] \leq [1,1] = [0,a] \rightarrow [0,a] = \nu[0,1] \rightarrow \nu[0,a]$$

and

$$\nu([0,1] \rightarrow [a,a]) = \nu[a,a] = [0,a] \leq [1,1] = [0,a] \rightarrow [0,a] = \nu[0,1] \rightarrow \nu[a,a]$$

Consequently,  $\nu([\psi_1, \psi_2] \rightarrow [\omega_1, \omega_2]) \leq \nu[\psi_1, \psi_2] \rightarrow \nu[\omega_1, \omega_2]$ . So (TL7) holds.

For (TL9), we can obtain that (TL9) holds through verification. Therefore, we can conclude that  $C = (C, \rightarrow, \nu, \mu, [0,a], [0,1], [1,1])$  is a triangle bounded semiregular  $L$ -algebra.

**Example 2.5** If  $L = (\{0,a,1\}, \rightarrow_L, 0,1)$  is a bounded  $L$ -algebra, then  $D = (D, \rightarrow, \nu, \mu, [0,0], [a,$

$a], [1,1])$  is a triangle bounded  $L$ -algebra, in which,  $D = \{[0,0], [a,a], [1,1]\}$ ,  $\nu[0,0] = \mu[0,0] = \nu[a,a] = [0,0]$ , and  $\nu[1,1] = \mu[1,1] = \mu[a,a] = [1,1]$ . The implication operations defined on  $L$  and  $D$  are presented in Tables 9 and 10, respectively.

**Table 9** Cayley table for the implication operation of  $L$

$\rightarrow_L$	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

**Table 10** Cayley table for the implication operation of  $D$

$\rightarrow$	$[0,0]$	$[a,a]$	$[1,1]$
$[0,0]$	$[1,1]$	$[1,1]$	$[1,1]$
$[a,a]$	$[a,a]$	$[1,1]$	$[1,1]$
$[1,1]$	$[0,0]$	$[a,a]$	$[1,1]$

It is obviously that  $(D, \rightarrow, [0,0], [1,1])$  is a bounded  $L$ -algebra. We use the same method as in Example 2.2, and it can be concluded that  $D$  is a triangle bounded  $L$ -algebra and it is also a triangle semiregular  $L$ -algebra with negation.

In Example 2.5, we can find that  $u' = [a,a]' = [a,a] \rightarrow [0,0] = [a,a] = u$ . Moreover, since  $(\nu[0,0]')' = (\nu[1,1]')' = [1,1]' = [0,0] = \mu[0,0]$ ,  $(\nu[1,1]')' = (\nu[0,0]')' = [0,0]' = [1,1] = \mu[1,1]$ , and  $(\nu[a,a]')' = (\nu[a,a]')' = [a,a]' = [a,a] = \mu[a,a]$ , we can observe that  $\mu\varphi = (\nu\varphi)'$  for any  $\varphi \in D$ . Therefore, this example can be used to explain the existence of the triangle bounded  $L$ -algebra given in Proposition 2.1.

In Ref. [8], a semiregular  $L$ -algebra  $L$  with negation can form a lattice with meet and join operations, where  $\varphi \vee \omega = (\varphi \sim \rightarrow \omega \sim) \rightarrow \varphi$ ,  $\varphi \wedge \omega = ((\varphi \rightarrow \omega) \rightarrow \varphi) \sim$  for all  $\varphi, \omega \in L$ .

We consider that the triangle semiregular  $L$ -algebra  $(A \subseteq L \times L, \rightarrow, \nu, \mu, 0, u, 1)$  with negation is generated by a semiregular  $L$ -algebra  $L$  with negation. For any  $[\psi_1, \psi_2], [\omega_1, \omega_2] \in A \subseteq L \times L$ , in which  $\psi_1 \leq \psi_2$ ,  $\omega_1 \leq \omega_2$ , and we define  $[\psi_1, \psi_2] \leq [\omega_1, \omega_2] \Leftrightarrow \psi_1 \leq \omega_1$  and  $\psi_2 \leq \omega_2$ . And the operations  $\nu$  and  $\mu$  are preserving meet and join operations for components, that is  $\nu[\psi_1, \psi_2] \vee \nu[\omega_1, \omega_2] = \nu[\psi_1 \vee \omega_1, \psi_2 \vee \omega_2]$ ,  $\nu[\psi_1, \psi_2] \wedge \nu[\omega_1, \omega_2] = \nu[\psi_1 \wedge \omega_1, \psi_2 \wedge \omega_2]$ ,  $\mu[\psi_1, \psi_2] \wedge \mu[\omega_1, \omega_2] = \mu[\psi_1 \wedge \omega_1, \psi_2 \wedge \omega_2]$ , and  $\mu[\psi_1, \psi_2] \vee \mu[\omega_1, \omega_2] = \mu[\psi_1 \vee \omega_1, \psi_2 \vee \omega_2]$ .

**Theorem 2.1** If a semiregular  $L$ -algebra  $L$  with

negation can generate a triangle semiregular  $L$ -algebra  $(A, \rightarrow, \nu, \mu, [0,0], [0,1], [1,1])$  with negation, then  $\nu[\psi_1, \psi_2] = [\psi_1, \psi_1]$  and  $\mu[\psi_1, \psi_2] = [\psi_2, \psi_2]$  for any  $[\psi_1, \psi_2] \in A$ .

**Proof** We first prove that  $\nu[\psi_1, \psi_2] = \nu[\psi_1, 1]$  for every  $[\psi_1, \psi_2] \in A$ . We can conclude that

$$\begin{aligned} \nu[\psi_1, \psi_2] &= \nu[\psi_1, \psi_2] \vee [0,0] = \\ \nu[\psi_1, \psi_2] \vee \nu[0,1] &= \nu[\psi_1, 1] \end{aligned}$$

Suppose  $\nu[\psi_1, \psi_2] = [\omega_1, \omega_2]$  and  $[\omega_1, \omega_2] \neq [\psi_1, \psi_1]$  for  $[\omega_1, \omega_2] \in A$ . Then  $[\omega_1, \omega_2] = \nu[\psi_1, \psi_2] = \nu[\psi_1, 1] = \nu[\psi_1, \psi_1] \leq [\psi_1, \psi_1]$ . We have  $\omega_1 < \psi_1$ , since  $\omega_1 = \psi_1$ , then  $\psi_1 = \omega_1 \leq \omega_2 \leq \psi_1$ , this implies that  $[\omega_1, \omega_2] = [\psi_1, \psi_1]$ , which is a contradiction.

We can conclude that  $\nu[\psi_1, 1] = \nu[\psi_1, \psi_2] = \nu\nu[\psi_1, \psi_2] = \nu[\omega_1, \omega_2] = \nu[\omega_1, 1]$ , and  $[1,1] = \mu[0,1] \leq \mu[\omega_1, 1] \leq \mu[\psi_1, 1]$ , since  $[0,1] \leq [\omega_1, 1] < [\psi_1, 1]$ . Therefore,  $\nu[\psi_1, 1] = \nu[\omega_1, 1]$  and  $\mu[\psi_1, 1] = \mu[\omega_1, 1]$ , we have  $[\psi_1, 1] = [\omega_1, 1]$ . Then  $\psi_1 = \omega_1$ . Therefore,  $\nu[\psi_1, \psi_2] = [\psi_1, \psi_1]$  for every  $[\psi_1, \psi_2]$  in  $A$ .

Now, we prove that  $\mu[\psi_1, \psi_2] = [\psi_2, \psi_2]$  for every  $[\psi_1, \psi_2]$  in  $A$ . We can get that

$$\begin{aligned} \mu[\psi_1, \psi_2] &= \mu[\psi_1, \psi_2] \wedge [1,1] = \\ \mu[\psi_1, \psi_2] \wedge \mu[0,1] &= \mu[0, \psi_2] \end{aligned}$$

for every  $[\psi_1, \psi_2] \in A$ . Suppose  $\mu[\psi_1, \psi_2] = [c_1, c_2]$  and  $[c_1, c_2] \neq [\psi_2, \psi_2]$ , for  $[c_1, c_2]$  in  $A$ .  $[c_1, c_2] = \mu[\psi_1, \psi_2] = \mu[0, \psi_2] = \mu[\psi_2, \psi_2] \geq [\psi_2, \psi_2]$ , we have  $c_2 > \psi_2$ . If  $c_2 = \psi_2$ , then  $c_1 \leq c_2 = \psi_2 \leq c_1$ , which would imply  $[c_1, c_2] = [\psi_2, \psi_2]$ , a contradiction.

We can conclude that  $\mu[0, \psi_2] = \mu[\psi_1, \psi_2] = \mu\mu[\psi_1, \psi_2] = \mu[c_1, c_2] = \mu[0, c_2]$ , and  $[0,0] = \nu[0, 1] \geq \nu[0, c_2] \geq \nu[0, \psi_2]$ , since  $[0,1] \geq [0, c_2] \geq [0, \psi_2]$ . Hence,  $\nu[0, \psi_2] = \nu[0, c_2]$  and  $\mu[0, \psi_2] = \mu[0, c_2]$ , we have  $[0, \psi_2] = [0, c_2]$ . Then  $\psi_2 = c_2$ . Therefore,  $\mu[\psi_1, \psi_2] = [\psi_2, \psi_2]$  for every  $[\psi_1, \psi_2]$  in  $A$ .

Therefore,  $\nu[\psi_1, \psi_2] = [\psi_1, \psi_1]$  and  $\mu[\psi_1, \psi_2] = [\psi_2, \psi_2]$  for any  $[\psi_1, \psi_2] \in A$ .

It is worth noting that in Theorem 2.1, if  $[\omega_1, \omega_2] \in A$ , then we also require  $[\omega_1, 1], [0, \omega_2], [\omega_1, \omega_1], [\omega_2, \omega_2] \in A$ .

In an algebra  $X$  with join and meet operations,  $\psi \in X$  is called meet irreducible, if for any  $\psi_1, \psi_2 \in X$  such that  $\psi_1 \wedge \psi_2 = \psi$ , then  $\psi_1 = \psi$  or  $\psi_2 = \psi$ . Similarly,  $\psi \in X$  is called join irreducible if for any  $\psi_1, \psi_2 \in X$ ,  $\psi_1 \vee \psi_2 = \psi$ , then  $\psi_1 = \psi$  or  $\psi_2 = \psi$ .

**Proposition 2.4** In a triangle semiregular  $L$ -algebra  $(A, \rightarrow, \nu, \mu, [0,0], u, [1,1])$  with negation,

which is generated by a semiregular  $L$ -algebra  $L$  with negation. If  $[0,0]$  is meet irreducible and  $[1,1]$  is join irreducible, then  $u = [0,1]$  or  $u = [\psi, \psi]$  for some  $\psi \in L$ .

**Proof** Let  $u = [\psi_1, \psi_2]$  for some  $[\psi_1, \psi_2] \in A$ . If  $\psi_1 = \psi_2$ , then  $u = [\psi_1, \psi_1]$ . If  $\psi_1 < \psi_2$ , then  $u \neq [\psi_2, \psi_2]$  and  $u \neq [\psi_1, \psi_1]$ . We have  $\nu[\psi_2, \psi_2] \neq \nu u = [0,0]$  or  $\mu[\psi_1, \psi_1] \neq \mu u = [1,1]$ .

We can conclude  $\nu[\psi_2, \psi_2] > \nu[\psi_1, \psi_2] = \nu u = [0,0]$ , since  $[1,1] = \mu u = \mu[\psi_1, \psi_2] < \mu[\psi_2, \psi_2]$ , because  $\mu$  and  $\nu$  are increasing. We can use the same method to prove  $\mu[\psi_1, \psi_1] < [1,1]$ . Therefore, we can conclude that  $\nu[\psi_2, \psi_2] \neq [0,0]$  and  $\mu[\psi_1, \psi_1] \neq [1,1]$ . Since  $\nu[0, \psi_2] \leq \nu[\psi_1, \psi_2] = [0,0]$ , it follows that

$$\begin{aligned} \nu[0,1] \wedge \nu[\psi_2, \psi_2] &= \nu([0,1] \wedge \\ &[\psi_2, \psi_2]) = \nu[0, \psi_2] = [0,0] \end{aligned}$$

since  $[0,0]$  is meet irreducible. Therefore,  $\nu[0,1] = [0,0]$ .

Since

$$[1,1] = \mu[\psi_1, \psi_2] \leq \mu[\psi_1, 1] = \mu([0,1] \vee [\psi_1, \psi_1]) = \mu[0,1] \vee \mu[\psi_1, \psi_1]$$

and  $[1,1]$  is join irreducible, we can conclude that  $[1,1] = \mu[0,1]$ .

This means that  $\nu[0,1] = \nu u$  and  $\mu[0,1] = \mu u$ , so we obtain  $u = [0,1]$ .

**Corollary 2.2** If a linear semiregular  $L$ -algebra  $L$  with negation can generate a triangle semiregular  $L$ -algebra  $(A, \rightarrow, \nu, \mu, [0,0], u, [1,1])$  with negation, in which  $A \subseteq L \times L$  contains more than four elements, then  $u = [0,1]$ .

**Proof** We can find that  $(A, \rightarrow, [0,0], [1,1])$  is a linear semiregular  $L$ -algebra with negation,  $[0,0]$  is meet irreducible and  $[1,1]$  is join irreducible, according to Proposition 2.4, we can find that  $u = [\psi, \psi]$  or  $u = [0,1]$ , where  $0 < \psi < 1$  and  $\psi \in L$ .

Now, we want to prove that  $u = [\psi, \psi]$  does not hold. Since  $A$  has more than four elements, for  $\omega \in L \setminus \{0, \psi, 1\}$ ,  $0 < \psi < \omega < 1$  or  $0 < \omega < \psi < 1$ . When  $0 < \psi < \omega < 1$ , we have  $[1,1] = \mu[\psi, \psi] \leq \mu[\psi, \omega]$ . So  $\mu[\psi, \omega] = \mu[\psi, \psi]$ . For  $\nu[\psi, \omega]$ , we have  $\nu[\psi, \omega] \not\geq [0, \psi]$  or  $\nu[\psi, \omega] \leq [0, \psi]$ .  $\nu[\psi, \omega] \leq [0, \psi]$  does not hold, since  $[0,1] \leq \mu[0,1] = \mu[\psi, \psi] \wedge \mu[0,1] = \mu[0, \psi] \leq \mu\nu[\psi, \omega] = \nu[\psi, \omega] \leq [\psi, \omega]$ , and so  $\omega < 1$ . Hence,  $\nu[\psi, \omega] \not\geq [0, \psi]$ . Consequently,  $\nu[\psi, \omega] = \nu\nu[\psi, \omega] < \nu[0, \psi] \leq \nu[\psi, \psi] = [0,0]$ . Then  $\nu[\psi, \omega] = \nu[\psi, \psi]$ . Since  $\mu[\psi, \omega] = \mu[\psi, \psi]$ , we can get that  $[\psi, \omega] = [\psi,$

$\psi]$ , a contradiction.

We can use the same method to prove when  $0 < \omega < \psi < 1$ ,  $u = [\psi, \psi]$  does not hold. Therefore, we can conclude that  $u = [0, 1]$ .

**Example 2.6** We set  $L = (L, \rightarrow, \nu, \mu, [0, 0], [0, 1], [1, 1])$ , in which,  $L = \{[0, 0], [0, a], [0, 1], [1, 1]\}$ ,  $\nu[0, 0] = \mu[0, 0] = \nu[0, 1] = \nu[0, a] = [0, 0]$ ,  $\mu[0, a] = [0, a]$ ,  $\nu[1, 1] = \mu[1, 1] = \mu[0, 1] = [1, 1]$ .

The operation  $\rightarrow$  of  $L$  is shown in Table 11.

**Table 11** Cayley table for the implication operation of  $L$

$\rightarrow$	$[0, 0]$	$[0, a]$	$[0, 1]$	$[1, 1]$
$[0, 0]$	$[1, 1]$	$[1, 1]$	$[1, 1]$	$[1, 1]$
$[0, a]$	$[0, 1]$	$[1, 1]$	$[1, 1]$	$[1, 1]$
$[0, 1]$	$[0, a]$	$[0, a]$	$[1, 1]$	$[1, 1]$
$[1, 1]$	$[0, 0]$	$[0, a]$	$[0, 1]$	$[1, 1]$

It is obviously that  $L = (L, \rightarrow, [0, 0], [1, 1])$  is a linear bounded semiregular  $L$ -algebra with negation. Moreover, we can find that  $\psi_1 \leq \psi_2$  for any  $[\psi_1, \psi_2] \in L$ .

It is easily that we can verify (TL1)–(TL6), (TL8), and (TL1')–(TL5') hold.

For (TL7). We can find that  $\nu([0, 1] \rightarrow [0, a]) = \nu[0, a] = [0, 0] \leq [1, 1] = \nu[0, 1] \rightarrow \nu[0, a]$ ,  $\nu([0, 1] \rightarrow [0, 0]) = \nu[0, a] = [0, 0] \leq [1, 1] = \nu[0, 1] \rightarrow \nu[0, 0]$ , and  $\nu([0, a] \rightarrow [0, 0]) = \nu[0, 1] = [0, 0] \leq [1, 1] = \nu[0, a] \rightarrow \nu[0, 0]$ . The same methods is used to prove that  $\nu([\psi_1, \psi_2] \rightarrow [\omega_1, \omega_2]) \leq \nu[\psi_1, \psi_2] \rightarrow \nu[\omega_1, \omega_2]$  for any  $[\psi_1, \psi_2], [\omega_1, \omega_2] \in L$ . So (TL7) holds.

For (TL9), we can obtain that (TL9) holds through verification.

Therefore, we can conclude that  $L = (L, \rightarrow, \nu, \mu, [0, a], [0, 1], [1, 1])$  is a linear triangle bounded semiregular  $L$ -algebra.

### 3 Triangle Ideals of Triangle Bounded $L$ -algebras

In this section, we propose and explore triangle ideals of different types of triangle bounded  $L$ -algebras.

**Definition 3.1** The subset  $I$  of triangle bounded  $L$ -algebra  $L$  is referred to an triangle ideal if for any  $\varphi, \psi \in L$ , the following conditions are satisfied:

- 1)  $1 \in I$ ;
- 2) If  $\varphi, \varphi \rightarrow \psi \in I$ , then  $\psi \in I$ ;
- 3) If  $\varphi \in I$ , then  $(\varphi \rightarrow \psi) \rightarrow \psi \in I$ ;
- 4) If  $\varphi \in I$ , then  $\psi \rightarrow \varphi, \psi \rightarrow (\varphi \rightarrow \psi) \in I$ ;
- 5) If  $\varphi \in I$ , then  $\nu\varphi \in I$ .

In a triangle ideal  $I$ , we have  $\mu\varphi \in I$  if  $\varphi \in I$ ,

since  $\varphi \leq \mu\varphi$  and Definition 3.1. We can easily discover that the triangle ideals of triangle bounded  $L$ -algebra  $L$  is special ideals of  $L$ -algebra, and  $\{1\}$  and  $L$  are two triangle ideals and they are trivial. It can be observed that when  $L$  can constitutes a triangle bounded  $KL$ -algebra, the set  $I$  is a triangle ideal if it only needs to satisfy conditions 1), 2), 3), and 5). If  $L$  can constitutes a triangle bounded  $CL$ -algebra, then  $I$  is a triangle ideal if it only needs to satisfy conditions 1), 2), and 5).

**Example 3.1** In Example 2.2,  $I = \{[a, a], [1, 1]\}$  is fulfilled with the conditions of the triangle ideal of a triangle bounded  $L$ -algebra. Thus, it is a triangle ideal of the triangle bounded  $L$ -algebra  $A$ .

**Proposition 3.1** Let  $(L, \rightarrow, \nu, \mu, 0, u, 1)$  be a triangle bounded  $L$ -algebra. If  $I$  is a triangle ideal of the triangle bounded  $L$ -algebra  $L$ , then  $\varphi \in I \Leftrightarrow \nu\varphi \in I$  and  $I \cap E(L)$  is an ideal of the bounded  $L$ -algebra  $E(L)$ .

**Proof** Let  $I$  be a triangle ideal of the triangle bounded  $L$ -algebra  $L$ . We have  $\varphi \in I \Leftrightarrow \nu\varphi \in I$ , according to Definition 3.1 and (TL1). It can be obtained that  $E(L)$  is a bounded  $L$ -algebra, according to Corollary 2.1. It is obvious that  $E(L)$  is a trivial ideal of  $E(L)$ . We can examine that  $I \cap E(L)$  is satisfied with conditions of ideal of  $L$ -algebra, for any  $\varphi, \omega \in E(L) \subseteq L$ .

It is straightforward to examine that  $1 \in I \cap E(L)$ . If  $\varphi, \varphi \rightarrow \omega \in I \cap E(L)$ , then  $\varphi, \varphi \rightarrow \omega \in I$  and  $\varphi, \varphi \rightarrow \omega \in E(L)$ . Since  $I$  is a triangle ideal of  $L$  and  $E(L)$  is an ideal of  $E(L)$ ,  $\omega \in I$  and  $\omega \in E(L)$ . Therefore,  $\omega \in I \cap E(L)$ .

If  $\varphi \in I \cap E(L)$ , then  $\varphi \in I$  and  $\varphi \in E(L)$ . We can get that  $(\varphi \rightarrow \omega) \rightarrow \omega \in I$  and  $\omega \rightarrow \varphi, \omega \rightarrow (\varphi \rightarrow \omega) \in I$ , since  $I$  is a triangle ideal of  $L$ . Moreover, we have  $(\varphi \rightarrow \omega) \rightarrow \omega \in E(L)$  and  $\omega \rightarrow \varphi, \omega \rightarrow (\varphi \rightarrow \omega) \in E(L)$ , since  $E(L)$  is a bounded  $L$ -algebra. Therefore,  $(\varphi \rightarrow \omega) \rightarrow \omega \in I \cap E(L)$  and  $\omega \rightarrow \varphi, \omega \rightarrow (\varphi \rightarrow \omega) \in I \cap E(L)$ .

Therefore,  $I \cap E(L)$  is an ideal of  $E(L)$ .

**Proposition 3.2** Let  $(L, \rightarrow, \nu, \mu, 0, u, 1)$  be a triangle bounded  $CL$ -algebra. If  $\varphi \in I \Leftrightarrow \nu\varphi \in I$  and  $I \cap E(L)$  is an ideal of the bounded  $L$ -algebra  $E(L)$ , then  $I \subseteq L$  is a triangle ideal of the triangle bounded  $CL$ -algebra  $L$ .

**Proof** If  $L$  is a triangle bounded  $CL$ -algebra and  $I \cap E(L)$  is an ideal of  $E(L)$ , and  $\varphi \in I \Leftrightarrow \nu\varphi \in I$ . Therefore, we only need to demonstrate that conditions 1) and 2) in Definition 3.1 are satisfied.

We can find that  $1 \in I \cap E(L) \subseteq I$ . For any  $\varphi, \omega \in L$ , if  $\varphi, \varphi \rightarrow \omega \in I$ , then  $\nu\varphi, \nu(\varphi \rightarrow \omega) \in I$ . Since  $\nu\varphi, \nu(\varphi \rightarrow \omega) \in E(L)$ ,  $\nu\varphi, \nu(\varphi \rightarrow \omega) \in I \cap E(L)$ . Because  $(\nu\varphi \rightarrow \nu\omega) \in E(L)$  and  $\nu(\varphi \rightarrow \omega) \rightarrow (\nu\varphi \rightarrow \nu\omega) = 1 \in I \cap E(L)$ , we can get that  $(\nu\varphi \rightarrow \nu\omega) \in I \cap E(L)$ . Because  $\nu\varphi \in I \cap E(L)$ , we have  $\nu\omega \in I \cap E(L) \subseteq I$ , and so  $\omega \in I$ .

Therefore,  $I \subseteq L$  is triangle ideal of the triangle bounded  $CL$ -algebra  $L$ .

**Theorem 3.1** If  $(L, \rightarrow, \nu, \mu, 0, u, 1)$  is a triangle bounded  $CL$ -algebra and  $I \subseteq L$ , then  $\varphi \in I \Leftrightarrow \nu\varphi \in I$  and  $I \cap E(L)$  is an ideal of the bounded  $L$ -algebra  $E(L)$  iff  $I$  is a triangle ideal of  $L$ .

**Proof** According to Propositions 3.1 and 3.2, we can get that  $\varphi \in I \Leftrightarrow \nu\varphi \in I$  and  $I \cap E(L)$  is a ideal of the bounded  $L$ -algebra  $E(L)$  iff  $I$  is a triangle ideal of the triangle bounded  $CL$ -algebra  $L$ .

**Definition 3.2** A bounded  $L$ -algebra  $L$  with negation is regarded as a Stonean bounded  $L$ -algebra if the supremum of  $\varphi'$  and  $\varphi''$  is 1, for any  $\varphi \in L$ .

**Definition 3.3** A triangle bounded  $L$ -algebra  $L$  is regarded as a Stonean triangle bounded  $L$ -algebra if  $L$  is a Stonean bounded  $L$ -algebra.

In Ref. [2], if  $L$  is a  $KL$ -algebra with negation and it satisfied condition  $\varphi \leq \omega$  iff  $\omega' \leq \varphi'$  for any  $\varphi, \omega \in L$ . Then  $L$  can define the meet and join operations.

**Example 3.2** Let

$$L = (\{ [0, a], [a, a], [0, 1], [a, 1] \}, \rightarrow, \nu, \mu, [0, a], [0, 1], [a, 1])$$

with Hasse diagram as shown in Fig.3, in which,  $0 \leq a \leq 1, \nu[0, 1] = \nu[0, a] = \mu[0, a] = \nu[a, a] = [0, a], \mu[a, a] = [a, a]$ , and  $\nu[a, 1] = \mu[a, 1] = \mu[0, 1] = [a, 1]$ . The operation  $\rightarrow$  defined on  $L$  is shown in Table 12.

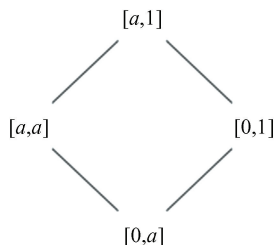


Fig.3 Hasse diagram of  $L$

Table 12 Cayley table for the implication operation of  $L$

$\rightarrow$	$[0, a]$	$[a, a]$	$[0, 1]$	$[a, 1]$
$[0, a]$	$[a, 1]$	$[a, 1]$	$[a, 1]$	$[a, 1]$
$[a, a]$	$[0, 1]$	$[a, 1]$	$[0, 1]$	$[a, 1]$
$[0, 1]$	$[a, a]$	$[a, a]$	$[a, 1]$	$[a, 1]$
$[a, 1]$	$[0, a]$	$[a, a]$	$[0, 1]$	$[a, 1]$

It is obviously that

$$L = (\{ [0, a], [a, a], [0, 1], [a, 1] \}, \rightarrow, \nu, \mu, [0, a], [0, 1], [a, 1])$$

It is a triangle bounded  $KL$ -algebra with negation and it satisfied condition  $x \leq y$  iff  $y' \leq x'$ , for all  $x, y \in L$ . Moreover, we find  $[0, a]' \vee [0, a]'' = [a, 1] \vee [0, a] = [a, 1]$ ,  $[a, a]' \vee [a, a]'' = [0, 1] \vee [a, a] = [a, 1]$ ,  $[0, 1]' \vee [0, 1]'' = [a, a] \vee [1, 1] = [a, 1]$ , and  $[a, 1]' \vee [a, 1]'' = [0, a] \vee [a, 1] = [a, 1]$ . Therefore,  $[\varrho_1, \varrho_2]' \vee [\varrho_1, \varrho_2]'' = [a, 1]$ , for any  $[\varrho_1, \varrho_2] \in C$ .

It can be concluded that  $L$  is a Stonean triangle bounded  $L$ -algebra.

**Example 3.3** Let  $L = (L, \rightarrow, \nu, \mu, [0, 0], [a, a], [1, 1])$  be a triangle bounded  $L$ -algebra with Hasse diagram as shown in Fig.4, in which,  $L = \{ [0, 0], [a, a], [1, 1] \}$ ,  $\nu[0, 0] = \mu[0, 0] = \nu[a, a] = [0, 0]$ ,  $\nu[1, 1] = \mu[1, 1] = \mu[a, a] = [1, 1]$ . The operation  $\rightarrow$  of  $L$  is presented in Table 13.

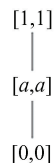


Fig.4 Hasse diagram of  $L$

Table 13 Cayley table for the implication operation of  $L$

$\rightarrow$	$[0, 0]$	$[a, a]$	$[1, 1]$
$[0, 0]$	$[1, 1]$	$[1, 1]$	$[1, 1]$
$[a, a]$	$[a, a]$	$[1, 1]$	$[1, 1]$
$[1, 1]$	$[0, 0]$	$[a, a]$	$[1, 1]$

It is obviously that  $L$  is a triangle bounded  $L$ -algebra. However, the supremum of  $[a, a]' = [a, a]$  and  $[a, a]'' = [a, a]$  is  $[a, a]$ , not  $[1, 1]$ .

Therefore,  $L$  is not a Stonean triangle bounded  $L$ -algebra.

**Proposition 3.3** If  $L$  is a Stonean triangle bounded  $KL$ -algebra, then  $E(L) = \{ \varphi \in L \mid \nu\varphi = \varphi \}$  and  $S = \{ \varphi \in L \mid (\nu\varphi)' = 0 \}$  are Stonean bounded  $KL$ -algebras and are closed with respect to the operations  $\nu$  and  $\mu$  on  $L$ .

**Proof** Since  $L$  is a Stonean triangle bounded  $KL$ -algebra,  $\varphi \rightarrow \omega = \nu\varphi \rightarrow \nu\omega = \nu(\nu\varphi \rightarrow \nu\omega) = \nu(\varphi \rightarrow \omega)$ , for any  $\varphi, \omega \in E(L) \subseteq L$ . Therefore,  $E(L) = \{ \varphi \in L \mid \nu\varphi = \varphi \}$  is closed under the operation  $\rightarrow$  in  $L$ . For any  $\varphi \in E(L)$ , we have  $\nu\varphi = \varphi \in E(L)$  and  $\mu\varphi = \varphi \in$

$E(L)$ , according to Corollary 2.1. Consequently, we can get that  $E(L) = \{\varphi \in L \mid \nu\varphi = \varphi\}$  is a Stonean bounded  $KL$ -algebra and is closed under the operations  $\nu$  and  $\mu$ .

According to Proposition 1.1 and  $\omega \leq \varphi \rightarrow \omega$ , we can find that  $(\nu(\varphi \rightarrow \omega))' \leq (\nu\omega)' = 0$ , for every  $\varphi, \omega \in S \subseteq L$ . Therefore, it can be concluded that  $\varphi \rightarrow \omega \in S$  and  $S = \{\varphi \in L \mid (\nu\varphi)' = 0\}$  is closed under the operation  $\rightarrow$  in  $L$ . Moreover,  $S = \{\varphi \in L \mid (\nu\varphi)' = 0\}$  is a Stonean bounded  $KL$ -algebra. For any  $\varphi \in S$ , we have  $(\nu\nu\varphi)' = (\nu\varphi)' = 0$ , since  $\nu\nu\varphi = \nu\varphi$  and  $(\nu\varphi)' = 0$ . Therefore,  $\nu\varphi \in S$ . Moreover,  $\nu\varphi \leq \nu\mu\varphi$ , since  $\varphi \leq \mu\varphi$ . Therefore,  $(\nu\mu\varphi)' \leq (\nu\varphi)' = 0$ . We can conclude that  $(\nu\mu\varphi)' = 0$ . Consequently  $\mu\varphi \in S$ . As a result,  $S = \{\varphi \in L \mid (\nu\varphi)' = 0\}$  is closed under the operations  $\nu$  and  $\mu$  on  $L$ .

**Proposition 3.4** In a triangle bounded  $KL$ -algebra  $L$  with negation, We will demonstrate that the following conclusions hold and are equivalent.

- 1)  $(\nu(\varphi \rightarrow \omega))' = (\nu(\omega \rightarrow \varphi))'$  for any  $\varphi, \omega \in L \setminus \{0\}$ .
- 2)  $(\nu\omega \rightarrow \nu\varphi)' = (\nu\varphi \rightarrow \nu\omega)'$  for any  $\varphi, \omega \in L \setminus \{0\}$ .
- 3)  $\{\varphi \in L \mid (\nu\varphi)' = 0\} = L \setminus \{0\}$ .

**Proof** 1)  $\Rightarrow$  3) : For every  $\varphi, \omega \in L \setminus \{0\}$ , we can get that  $(\nu(\omega \rightarrow \varphi))' = (\nu(\varphi \rightarrow \omega))'$ . When  $\omega = 1$ ,  $(\nu(1 \rightarrow \varphi))' = (\nu(\varphi \rightarrow 1))'$ , that is  $(\nu\varphi)' = (\nu 1)' = 1' = 0$ . It can be concluded that  $L \setminus \{0\} \subseteq \{\varphi \in L \mid (\nu\varphi)' = 0\}$ . Moreover, it is obvious that  $\{\varphi \in L \mid (\nu\varphi)' = 0\} \subseteq L \setminus \{0\}$ . Therefore,  $\{\varphi \in L \mid (\nu\varphi)' = 0\} = L \setminus \{0\}$ .

3)  $\Rightarrow$  1) : For any  $\varphi, \omega \in L \setminus \{0\}$ , if  $\{\varphi \in L \mid (\nu\varphi)' = 0\} = L \setminus \{0\}$ , then  $\varphi, \omega \in \{\varphi \in L \mid (\nu\varphi)' = 0\}$ . We have  $(\nu(\varphi \rightarrow \omega))' \leq (\nu\omega)' = 0$ , since  $\omega \leq \varphi \rightarrow \omega$  and Proposition 1.1. So,  $(\nu(\varphi \rightarrow \omega))' = 0$ . We can use the same method to obtain  $(\nu(\omega \rightarrow \varphi))' = 0$ . Therefore,  $(\nu(\varphi \rightarrow \omega))' = (\nu(\omega \rightarrow \varphi))'$  for any  $\varphi, \omega \in L \setminus \{0\}$ .

2)  $\Rightarrow$  3) : Let  $(\nu\omega \rightarrow \nu\varphi)' = (\nu\varphi \rightarrow \nu\omega)'$  for any  $\varphi, \omega \in L \setminus \{0\}$ . If  $\omega = 1$ , then

$$\begin{aligned} (\nu\omega \rightarrow \nu\varphi)' &= (\nu\varphi \rightarrow \nu\omega)' \Rightarrow \\ (\nu 1 \rightarrow \nu\varphi)' &= (\nu\varphi \rightarrow \nu 1)' \Rightarrow \\ (\nu\varphi)' &= 1' = 0 \end{aligned}$$

We can conclude that  $L \setminus \{0\} \subseteq \{\varphi \in L \mid (\nu\varphi)' = 0\}$ . Moreover, it is obvious that  $\{\varphi \in L \mid (\nu\varphi)' = 0\} \subseteq L \setminus \{0\}$ . Therefore,  $\{\varphi \in L \mid (\nu\varphi)' = 0\} = L \setminus \{0\}$ .

3)  $\Rightarrow$  2) : Let  $\{\varphi \in L \mid (\nu\varphi)' = 0\} = L \setminus \{0\}$ . Then for any  $\varphi, \omega \in L \setminus \{0\} = \{\varphi \in L \mid (\nu\varphi)' = 0\}$ ,

we have  $(\nu\omega \rightarrow \nu\varphi)' \leq (\nu\varphi)' = 0$ , since  $\nu\varphi \leq \nu\omega \rightarrow \nu\varphi$  and Proposition 1.1. Therefore,  $(\nu\omega \rightarrow \nu\varphi)' = 0$ . We can use the same method to obtain  $(\nu\varphi \rightarrow \nu\omega)' = 0$ . Therefore,  $(\nu\omega \rightarrow \nu\varphi)' = (\nu\varphi \rightarrow \nu\omega)'$  for any  $\varphi, \omega \in L \setminus \{0\}$ .

**Theorem 3.2** If the triangle bounded  $KL$ -algebra  $L$  is satisfied with condition  $(\nu\omega \rightarrow \nu\varphi)' = (\nu\varphi \rightarrow \nu\omega)'$ , for any  $\varphi, \omega \in L \setminus \{0\}$ , then  $L$  is a Stonean triangle bounded  $L$ -algebra.

**Proof** According to Proposition 3.4, if  $(\nu\varphi \rightarrow \nu\omega)' = (\nu\omega \rightarrow \nu\varphi)'$ , for any  $\varphi, \omega \in L \setminus \{0\}$ , then  $\{\varphi \in L \mid (\nu\varphi)' = 0\} = L \setminus \{0\}$ . We can conclude that  $\varphi' \leq (\nu\varphi)' = 0$ , since (TL1) and Proposition 1.1. Hence,  $\varphi' = 0$  and so  $\varphi'' = 1$ . Therefore, the supremum of  $\varphi'$  and  $\varphi''$  is 1, for any  $\varphi \in L$ . Then  $L$  is a Stonean triangle bounded  $L$ -algebra.

When  $L$  is a Stonean triangle bounded  $L$ -algebra and  $L' = \{\varphi' \mid \varphi \in L\}$  is a complement space of  $L$ . Then we can observe the supremum of  $\varphi$  and  $\varphi'$  is 1, for any  $\varphi \in L'$ . If the infimum of  $\varphi$  and  $\varphi'$  is 0, for any  $\varphi \in L'$ , then  $L'$  forms a complementary lattice. In the following text, we will refer to Stonean triangle bounded  $L$ -algebra as the Stonean algebra, without causing any ambiguity.

**Corollary 3.1** If the triangle bounded  $KL$ -algebra  $L$  is satisfied with one of the conditions in Proposition 3.4, then  $L$  is a Stonean algebra.

**Definition 3.4** In a triangle bounded  $L$ -algebra  $L$  with negation, the triangle ideal  $I$  of  $L$  is a Stonean triangle ideal if the supremum of  $\varphi'$  and  $\varphi''$  belongs to  $I$  under  $\nu$  operation, for any  $\varphi \in L$ .

**Definition 3.5** In a triangle bounded  $L$ -algebra  $L$  with negation, the triangle ideal  $I$  is an extended Stonean triangle ideal if the supremum of  $(\nu\varphi)'$  and  $(\nu\varphi)''$  belongs to  $I$ , for any  $\varphi \in L$ .

**Remark 3.1** Let  $L$  be a triangle bounded  $KL$ -algebra with negation and join operation. If  $I$  is a Stonean triangle ideal of  $L$  and  $\nu(\varphi \vee \omega) = \nu\varphi \vee \nu\omega$  for any  $\varphi, \omega \in L$ , then  $I$  is an extended Stonean triangle ideal.

**Proof** The supremum of  $\varphi'$  and  $\varphi''$  belongs to  $I$  under  $\nu$  operation, that is  $\nu(\varphi'' \vee \varphi') \in I$ , for all  $\varphi \in L$ , since  $I$  is a Stonean triangle ideal. Moreover,

$$\begin{aligned} \nu(\varphi'' \vee \varphi') &= \nu(\varphi') \vee \\ \nu(\varphi'') &\leq (\nu\varphi)'' \vee (\nu\varphi)' \end{aligned}$$

according to (TL7). Therefore,  $\nu(\varphi'' \vee \varphi') \rightarrow ((\nu\varphi)' \vee (\nu\varphi)'') = 1 \in I$ . Since  $\nu(\varphi'' \vee \varphi') \in I$ , we have  $(\nu\varphi)' \vee (\nu\varphi)'' \in I$ . Hence, we can get that

the supremum of  $(\nu\varphi)'$  and  $(\nu\varphi)''$  belongs to  $I$ , for any  $\varphi \in L$ . Therefore,  $I$  is an extended Stonean triangle ideal.

**Example 3.4** In Example 3.2, it is clearly that  $I = \{[a, 1]\}$  is fulfilled with the conditions of the ideal of a triangle bounded  $L$ -algebra. Thus, it is a triangle ideal of

$$L = (\{[0, a], [a, a], [0, 1], [a, 1]\}, \rightarrow, \nu, \mu, [0, a], [0, 1], [a, 1])$$

Moreover, we can find that  $\{[a, 1]\}$  is a Stonean triangle ideal of  $L$  and  $\{[a, 1]\}$  is also an extended Stonean triangle ideal.

**Example 3.5** Let  $L = (\{0, a, 1\}, \rightarrow, 0, a, 1)$  be a triangle bounded  $L$ -algebra, in which,  $\mu 0 = \nu 0 = \nu a = 0$  and  $\nu 1 = \mu 1 = \mu a = 1$ . The operation  $\rightarrow$  of  $L$  is presented in Table 14.

**Table 14** Cayley table for the implication operation of  $L$

$\rightarrow$	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

We can prove that  $I = \{1\}$  is an extended Stonean triangle ideal of  $L$ . However, we can find that the supremum of  $a' = a$  and  $a'' = a$  do not belong to  $I$ . Consequently,  $I = \{1\}$  is not a Stonean triangle ideal of  $L$ .

**Theorem 3.3** Let  $L$  be a Stonean triangle bounded  $L$ -algebra. If  $I$  is a triangle ideal of  $L$ , then  $I$  is a Stonean triangle ideal of  $L$ .

**Proof** When  $L$  is a Stonean triangle bounded  $L$ -algebra and  $I$  is a triangle ideal of  $L$ . For all  $\varphi \in L$ , The supremum of  $\varphi'$  and  $\varphi''$  is 1, and so the supremum of  $\varphi'$  and  $\varphi''$  is also 1 under  $\nu$  operation, since  $\nu 1 = 1$  and  $1 \in I$ . Consequently,  $I$  is a Stonean triangle ideal of  $L$ .

**Proposition 3.5** In a triangle bounded  $L$ -algebra  $L$  with negation. If  $\{1\}$  is a Stonean triangle ideal of  $L$ , then  $L$  is Stonean algebra.

**Proof** Let  $\{1\}$  be a triangle ideal of  $L$ . Then the supremum of  $\varphi'$  and  $\varphi''$  belongs to  $\{1\}$  under  $\nu$  operation, for any  $\varphi \in L$ . Since  $\nu\varphi \leq \varphi$  and  $\nu 1 = 1$ , it can be concluded that the supremum of  $\varphi'$  and  $\varphi''$  is 1. Hence,  $L$  is a Stonean triangle bounded  $L$ -algebra.

**Corollary 3.2** In a triangle bounded  $L$ -algebra  $L$  with negation,  $\{1\}$  is a Stonean triangle ideal of  $L$  iff  $L$  is Stonean algebra.

**Theorem 3.4** Let  $I$  be an extended Stonean triangle ideal of the triangle bounded  $L$ -algebra  $L$  and  $J$

be a triangle ideal of  $L$ , where  $I \subseteq J$ . Then  $J$  is an extended Stonean triangle ideal of  $L$ .

**Proof** Let  $I$  be an extended Stonean triangle ideal of  $L$ . Then the supremum of  $(\nu\varphi)'$  and  $(\nu\varphi)''$  belongs to  $I$ , for all  $\varphi \in L$ . So, it can be concluded that the supremum of  $(\nu\varphi)'$  and  $(\nu\varphi)''$  belongs to  $J$ , since  $I \subseteq J$ . Consequently,  $J$  is an extended Stonean triangle ideal in  $L$ .

**Corollary 3.3** Let  $L$  be a triangle bounded  $L$ -algebra with negation.  $\{1\}$  is an extended Stonean triangle ideal of  $L$  iff every triangle ideal of  $L$  is also an extended Stonean triangle ideal.

**Proposition 3.6** In a Stonean triangle bounded  $KL$ -algebra  $L$  with join operation. If  $\nu(\varphi \vee \omega) = \nu\varphi \vee \nu\omega$  for any  $\varphi, \omega \in L$ , then triangle ideal  $I$  of  $L$  is also an extended Stonean triangle ideal.

**Proof** Let  $L$  be a Stonean triangle bounded  $KL$ -algebra with join operation, then  $\{1\}$  is a Stonean triangle ideal of  $L$ , according to Corollary 3.2. We can conclude that  $\{1\}$  is an extended Stonean triangle ideal of  $L$  by Remark 3.1. Therefore, every triangle ideal  $I$  is an extended Stonean triangle ideal in  $L$ , according to Corollary 3.3.

**Proposition 3.7** If  $I$  is a triangle ideal of a triangle bounded  $KL$ -algebra  $L$  with double negation and if  $\varphi \in L$ , then  $\varphi \in I$  or  $\varphi' \in I$ , then  $I$  is an extended Stonean triangle ideal.

**Proof** For every  $\varphi \in L$ . When  $\varphi \in I$ , we have  $\varphi'' \in I$ , since  $\nu\varphi \in I$  and  $\nu\varphi = (\nu\varphi)''$ . Therefore, the supremum of  $(\nu\varphi)'$  and  $(\nu\varphi)''$  belongs to  $I$ . When  $\varphi \notin I$ , we have  $\varphi' \in I$ , and so  $(\nu\varphi)' \in L$ , since  $\varphi' \leq (\nu\varphi)'$ . Thus the supremum of  $(\nu\varphi)'$  and  $(\nu\varphi)''$  belongs to  $I$ . Consequently,  $I$  is an extended Stonean triangle ideal.

**Definition 3.6** For any  $\varphi, \omega \in L$ , if the subset  $I$  of a triangle bounded  $L$ -algebra  $L$  with the meet operation is referred to as a lattice ideal, the following conditions are satisfied:

- 1) If  $\varphi \in I$  and  $\varphi \leq \omega$ , then  $\omega \in I$ ;
- 2) If  $\varphi, \omega \in I$ , then  $\varphi \wedge \omega \in I$ ;
- 3) If  $\varphi \in I$ , then  $\nu\varphi \in I$ .

**Example 3.6** In Example 3.2, it is clearly that  $I = \{[a, 1]\}$  is fulfilled with the conditions of the lattice ideal. Thus,  $I = \{[a, 1]\}$  is also a lattice ideal.

**Definition 3.7**<sup>[2]</sup> We refer to  $(S, *, \rightarrow, 1)$  as a left hoop if the following conditions are satisfied for any  $\varphi, \psi, \omega \in S$ :

- 1)  $(S, *, 1)$  is a monoid;
- 2)  $\varphi \rightarrow \varphi = 1$ ;

$$3) (\varphi * \psi) \rightarrow \omega = \varphi \rightarrow (\varphi \rightarrow \omega);$$

$$4) (\psi \rightarrow \omega) * \psi = (\omega \rightarrow \psi) * \omega.$$

In Ref. [8], a semiregular  $L$ -algebra  $L$  with negation is a left hoop. Hence, we have  $\varphi * \omega \leq \varphi \wedge \omega$  for any  $\varphi, \omega \in L$ , since  $\varphi \leq \omega \rightarrow \varphi$  and  $\varphi \leq \omega \rightarrow \omega = 1$ .

**Proposition 3.8** If  $L$  is a triangle semiregular  $L$ -algebra with negation, then a triangle ideal of  $L$  is a lattice ideal.

**Proof** If  $I$  is a triangle ideal of a triangle semiregular  $L$ -algebra  $L$  with negation, then  $I$  holds on conditions 1) and 3) of Definition 3.6. We only need to prove that  $\varphi \wedge \omega \in I$  for any  $\varphi, \omega \in I$ . Since  $\varphi * \omega \leq \varphi \wedge \omega, \varphi \leq \omega \rightarrow \varphi \wedge \omega$ . So  $\varphi \rightarrow (\omega \rightarrow \varphi \wedge \omega) = 1 \in I$ . Since  $I$  is a triangle ideal,  $\omega \rightarrow \varphi \wedge \omega \in I$ . Hence,  $\varphi \wedge \omega \in I$ , since  $\omega \in I$ . Therefore,  $I$  can form a lattice ideal of  $L$ .

## 4 Conclusions

Triangle bounded  $L$ -algebras and ideals have a great significance in the study of fuzzy logics and logical algebras. This paper focuses on exploring triangle bounded  $L$ -algebras and triangle ideals. Firstly, we extend the definition of bounded  $L$ -algebra to the notion of triangle bounded  $L$ -algebras by adding a constant  $u \notin \{0, 1\}$  and two unary operations  $\nu$  and  $\mu$ . Secondly, we defined the notion of triangle ideals of triangle bounded  $L$ -algebras to explore the connection between the triangle ideal  $I$  of triangle bounded  $L$ -algebra  $L$  and the ideal  $I \cap E(L)$  of bounded  $L$ -algebra  $E(L)$ . In addition, by introducing the concept of Stonean triangle bounded  $L$ -algebra, we study its properties and the connection between Stonean triangle algebras and (extended) Stonean triangle ideals. Various classes of triangle ideals, including Stonean triangle ideals, extended Stonean triangle ideals, and lattice ideals, are introduced and studied. Finally, the interrelationships among various types of ideals are investigated. In the future, we will characterize the structure of triangle bounded  $L$ -algebras by studying closure operators, interior operators, and local bounded  $L$ -algebras, and further investigate closure operators and interior operators in Stonean spaces.

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